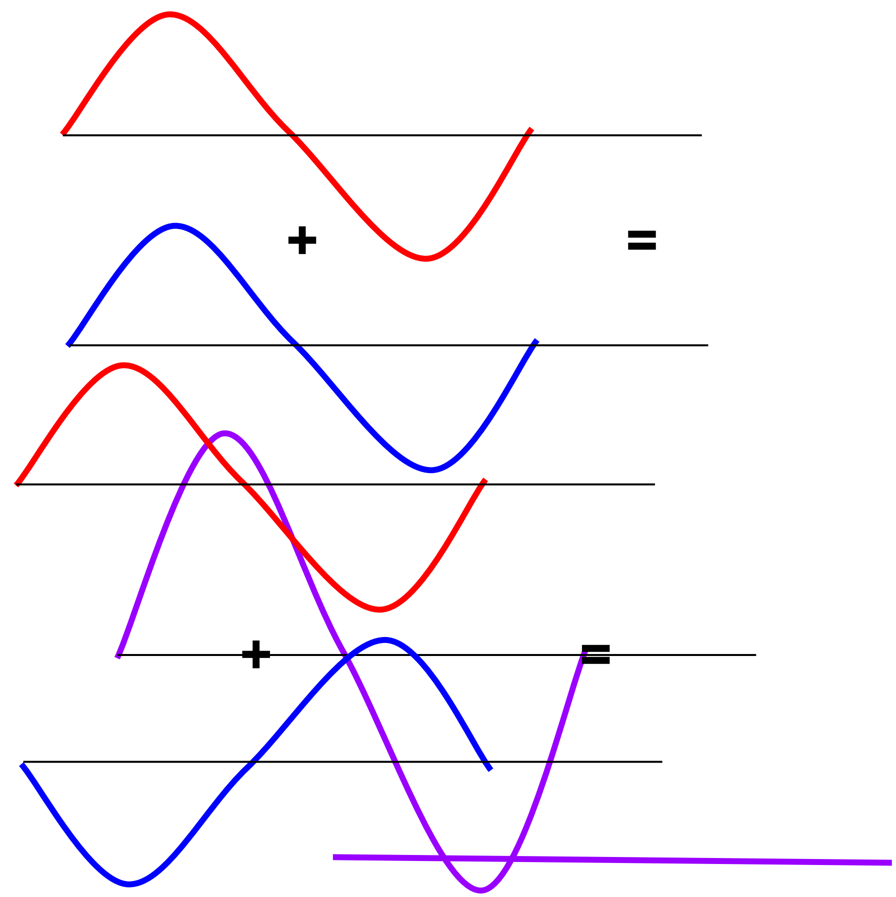
Working with i and rotations

Complex Numbers
in
Quantum Information Science

Quantum Interference

- Qubits in superposition constructively and destructively interfere due to phase differences
  - We refer to the phase difference between $|0\rangle$ and $|1\rangle$ as ***relative phase*** (ex. the -1 in $|\psi\rangle = |-\rangle = \alpha|0\rangle - \beta|1\rangle$)
- Think about waves:
  - **Like** signs in amplitude -> bigger combined wave
  - **Different** signs in amplitude -> combined amplitude goes to zero



Probability Amplitudes and Phase

- We have looked at qubit states with ***probability amplitudes*** that are ***real numbers***
    - Example real numbers: -1, 5.3, 7/8, $\sqrt{2}$, π
    - Phase differences occur when $|0\rangle$ and $|1\rangle$ have opposite sign
- Probability amplitudes associated with quantum state can also be ***complex numbers***

Qubit state:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Mathematical Symbols:

$\mathbb{R}$: Set of Real Numbers. Includes positive and negative values that are rational or irrational

$\mathbb{C}$: Set of Complex Numbers

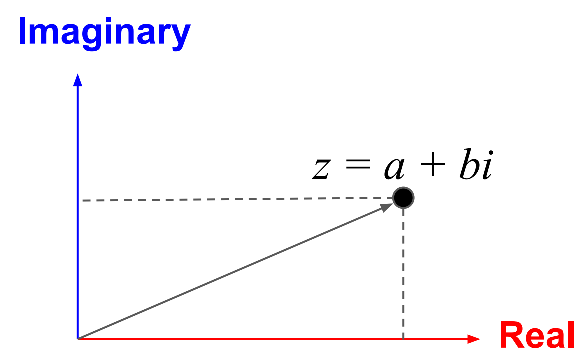
Complex Numbers

Complex numbers include a real and imaginary component

$$z = \boxed{a} + \boxed{bi}$$

Real Part

Imaginary Part



***i* is an imaginary number where $i = \sqrt{-1}$ and $i^2 = -1$**

Working with Complex Numbers

- Like terms combine for addition and subtraction
  - (3 + 4i) + (6 - 2i)
    = (9 + 2i)
  - (4) - (3 - i)
    = (1+i)
- When multiplying, apply FOIL technique
  - (3 - 2i) (1 +3i)
    First: 3*1 = 3
    Outside: 3*(3i) = 9i
    Inside: (-2i)*1 = -2i
    Last: (-2i)*(3i) = $-6i^2$ = 6

Combine!

(3 - 2i) (1 +3i) =
9 + 7i

Remember: i*i = i^2 = -1 and (-1)*(-1) = 1

PRACTICE: Multiply these two complex numbers.

$(3+i)(-2i)$

a. $-4i$
b. $-2-6i$
c. $2+6i$
d. $2-6i$

$(6-2i)(7-4i)$

a. $42 - 30i$
b. $50 - 38i$
c. $36 - 38i$
d. $42 + 8i$

PRACTICE: Multiply these two complex numbers.

(3+i)(-2i)

a. -4i
b. -2-6i
c. 2+6i
d. 2-6i

(6-2i)(7-4i)

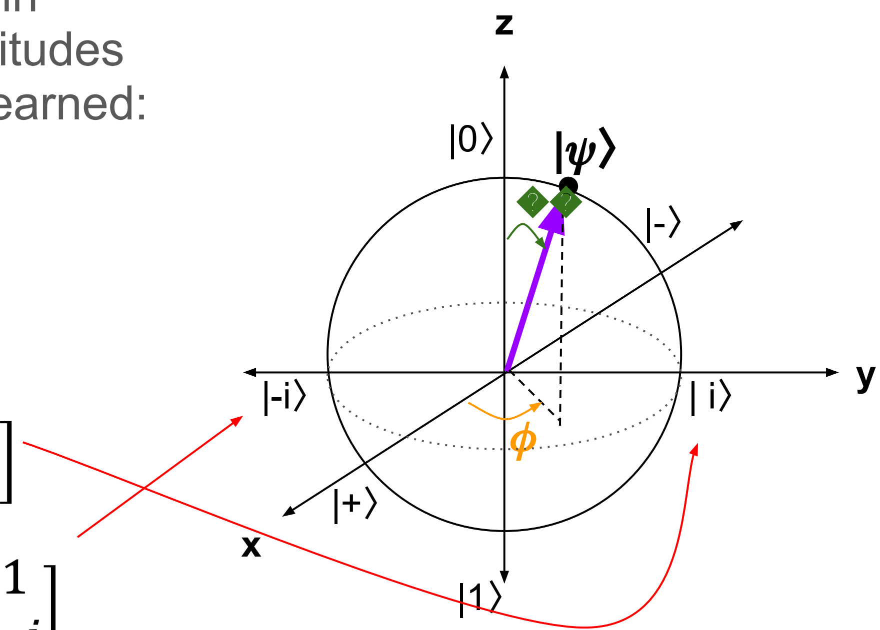
a. 42 - 30i
b. 50 - 38i
c. 34 - 38i
d. 42 + 8i

Complex values in quantum computing

- Changes in ϕ ($\neq 0, \pi$) can result in complex-valued probability amplitudes
- On the Bloch Sphere, we have learned:
  - $|0\rangle$ and $|1\rangle$ are on the z-axis
  - $|+\rangle$ and $|-\rangle$ are on the x-axis
- New states on the y-axis:

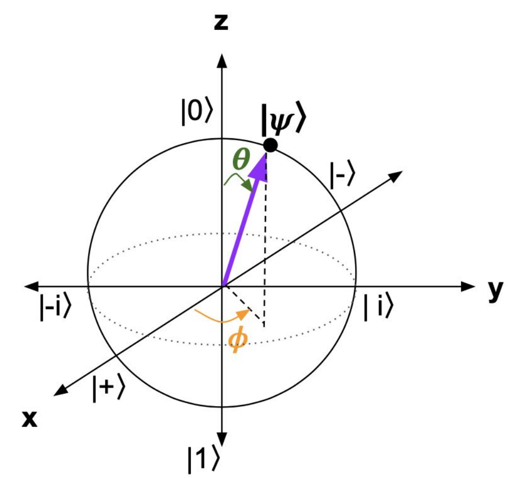
$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}$$

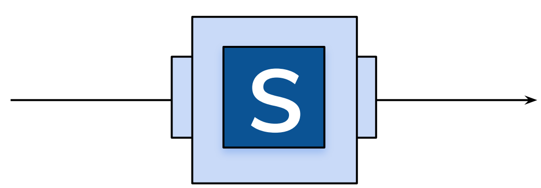
$$|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix}$$



Rotating Phase with S and T gates

Two new gates commonly used to rotate qubit phase (ϕ on Bloch Sphere):

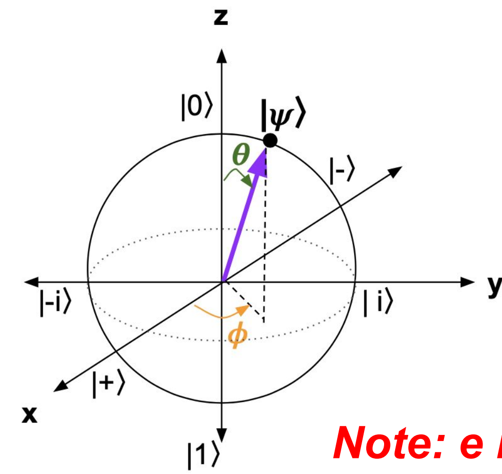


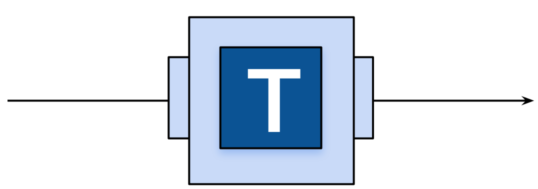


$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

Rotating Phase with S and T gates

Two new gates commonly used to rotate qubit phase (ϕ on Bloch Sphere):





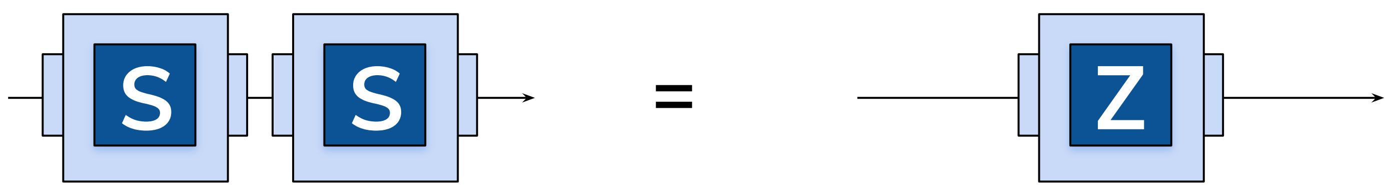
$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}$$

Note: e is the mathematical constant known as Euler's number (≈2.71828, the base of the natural logarithm)

Is S its own inverse?!?

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

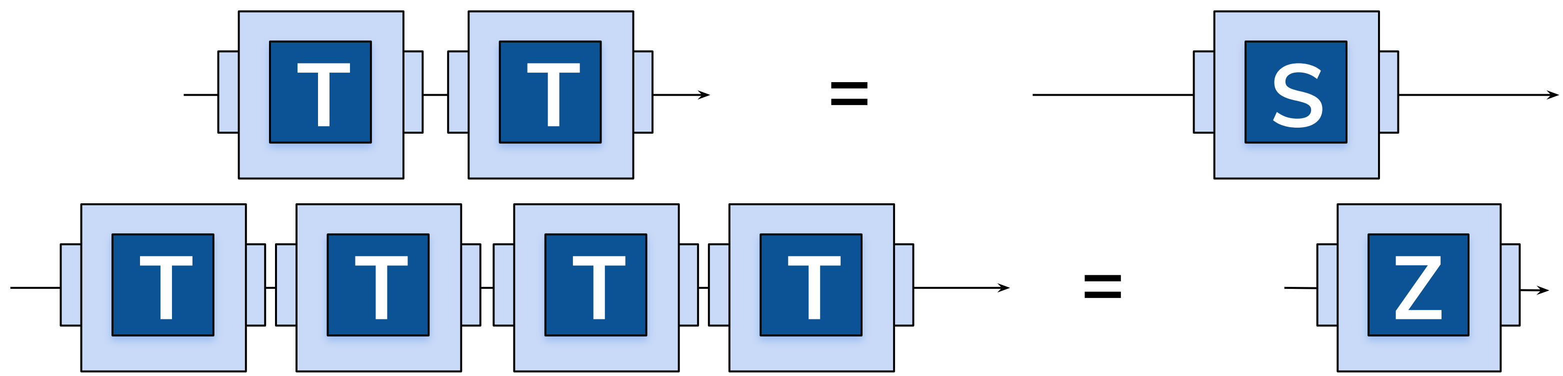
$$S^2 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} =$$



Is T its own inverse?!?

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}$$

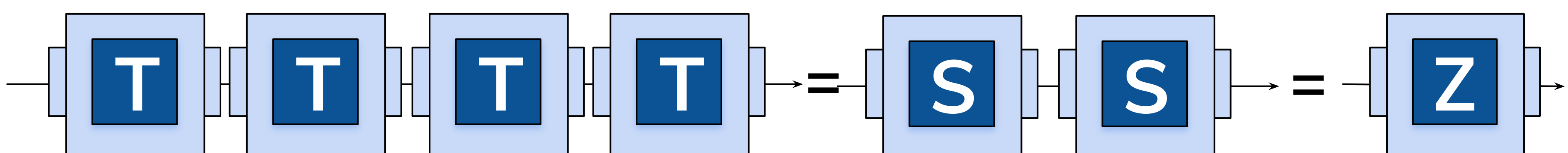
$$T^2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}$$



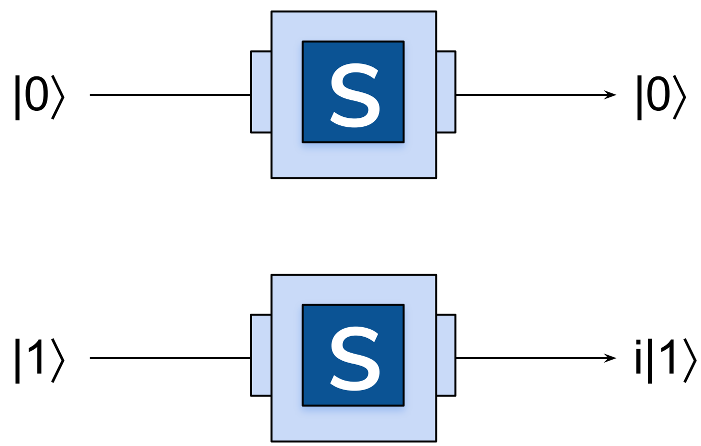
Relations between the Phase Gates: T, S, Z

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \sqrt{Z} \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix} = \sqrt{S} = \sqrt[4]{Z}$$

$$S^2 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z \quad T^2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = S$$



Example: State Transformation with S



$$S|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

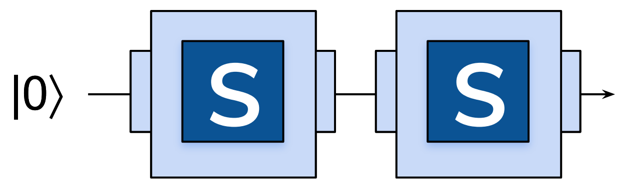
$$S|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix}$$

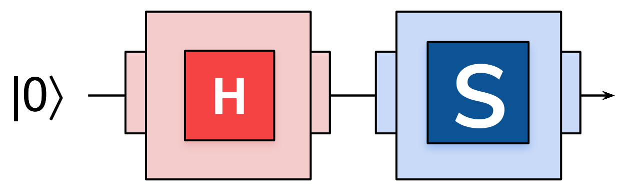
As with negative / positive phase, i / not-i phase is meaningless without superposition

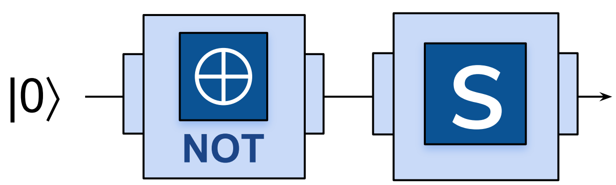
Why are Complex Values used in Quantum Computing? (SIMPLE ANSWER)

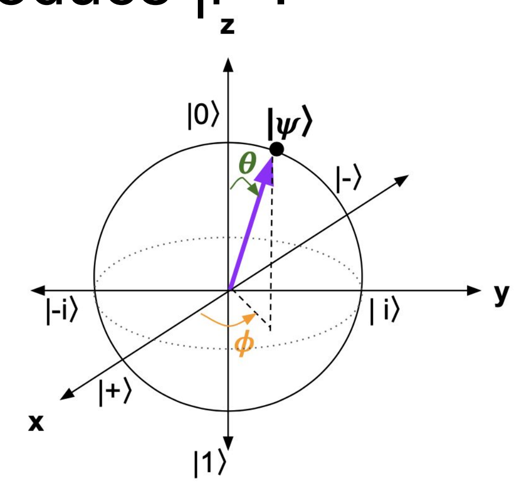
- Quantum signals must interfere with phases at varying degrees….critical for many quantum algorithms!
- While in superposition, value of quantum state is unknown and complex values describe unknown values (especially those that oscillate) very well
- In the end, physical meaning not connected to complex quantities, but operation that *produces* real numbers
  - Phase associated with probability amplitudes not measured
  - Requirement that for $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$

PRACTICE: What circuit allows for us to produce |i>?

A.


B.


C.




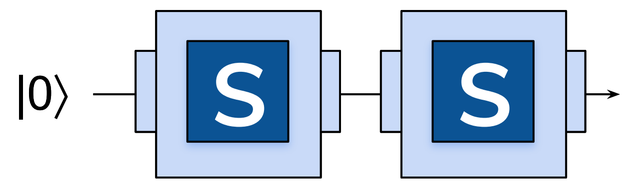
Hint: NOT = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

H = $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

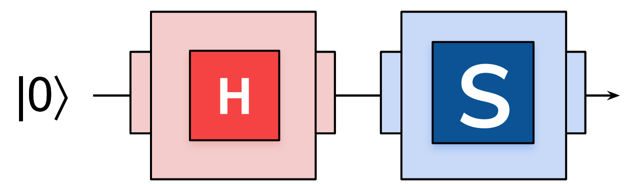
S = $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

PRACTICE: What circuit allows for us to produce |i>?

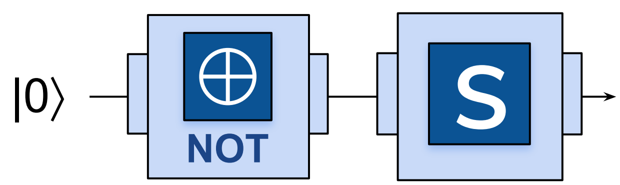
A.



B.



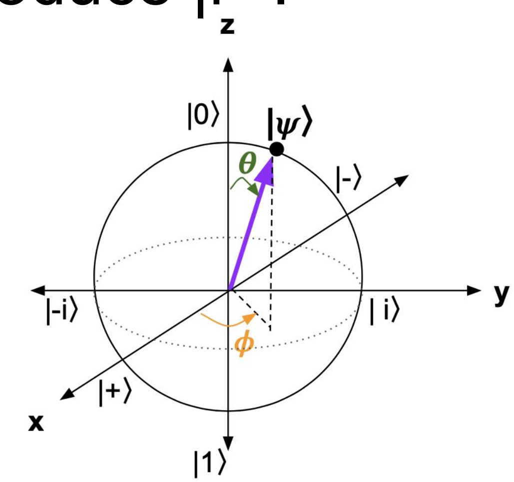
C.



Hint: NOT = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

H = $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

S = $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$



PRACTICE

What is the value of i^3?

a. i
b. -1
c. -i
d. 1

PRACTICE

What is the value of i^3?

a. i
b. -1
c. -i
d. 1

PRACTICE

What is the value of i^4?

a. i
b. -1
c. -i
d. 1

PRACTICE

What is the value of i^4?

a. i
b. -1
c. -i
d. 1

Absolute Value of a Complex Number

Square root of value times complex conjugate

Absolute value:

$$|z| = \sqrt{z\bar{z}}$$

Example: Absolute Value of a Complex Number

Absolute value:

$$|z| = \sqrt{z\bar{z}}$$

|3+4i| =

Sqrt ((3 + 4i)(3 - 4i)) =

Sqrt(9 + 12i -12i +16) =

Sqrt(25) =

5

|6 - 8i| =

Sqrt ((6 - 8i)(6 + 8i)) =

Sqrt(36 - 48i +48i +64) =

Sqrt(100) =

10

Example: Calculate Probabilities for |i⟩

Absolute value:

$$|z| = \sqrt{z\bar{z}}$$

Known: Probability for measuring $|0\rangle$ is $|\alpha|^2$ and $|1\rangle$ is $|\beta|^2$ for $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}$$

*Note: i*i = -1 , -i*i = 1*

Probability of $|0\rangle$ → $|\alpha|^2 = \left(\sqrt{\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

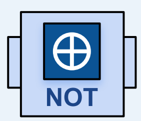
Probability of $|1\rangle$ → $|\beta|^2 = \left(\sqrt{\left(\frac{i}{\sqrt{2}}\right)\left(\frac{-i}{\sqrt{2}}\right)}\right)^2 = \left(\sqrt{\left(\frac{1}{2}\right)}\right)^2 = \frac{1}{2}$

The Bloch Sphere and Single-Qubit Rotation Operations

Review: Single-Qubit Operations

- Qubit state represented with the Bloch Sphere
- Qubit gates we have learned about include:

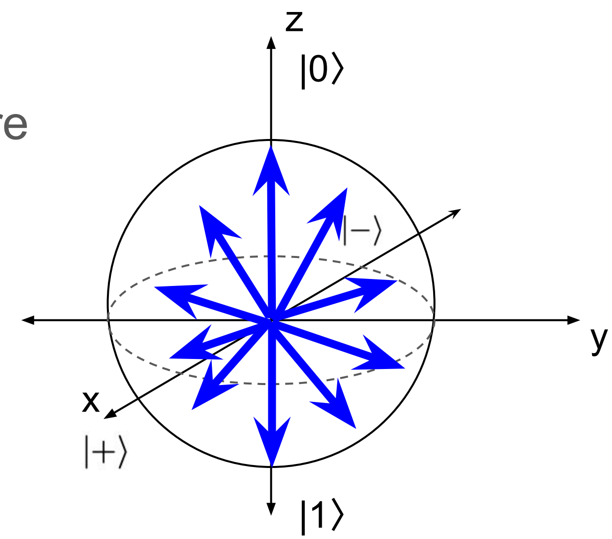
NOT



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$|1\rangle$ = NOT $|0\rangle$

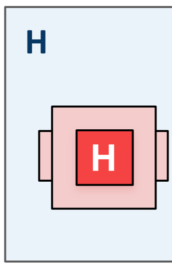
$|0\rangle$ = NOT $|1\rangle$



The NOT gate is also called the X gate...
It implements a rotation of π around the x-axis of the Bloch Sphere!

Review: Single-Qubit Operations

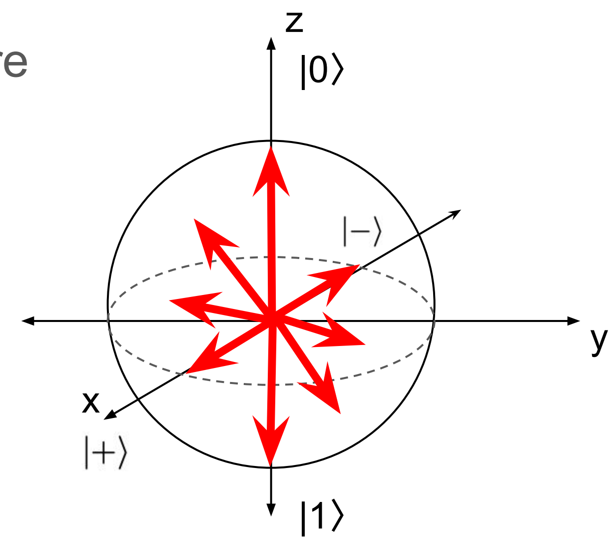
- Qubit state represented with the Bloch Sphere
- Qubit gates we have learned about include:



$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

|+⟩= 1/√2(|0⟩+|1⟩) = H |0⟩

|-⟩= 1/√2(|0⟩-|1⟩) = H |1⟩



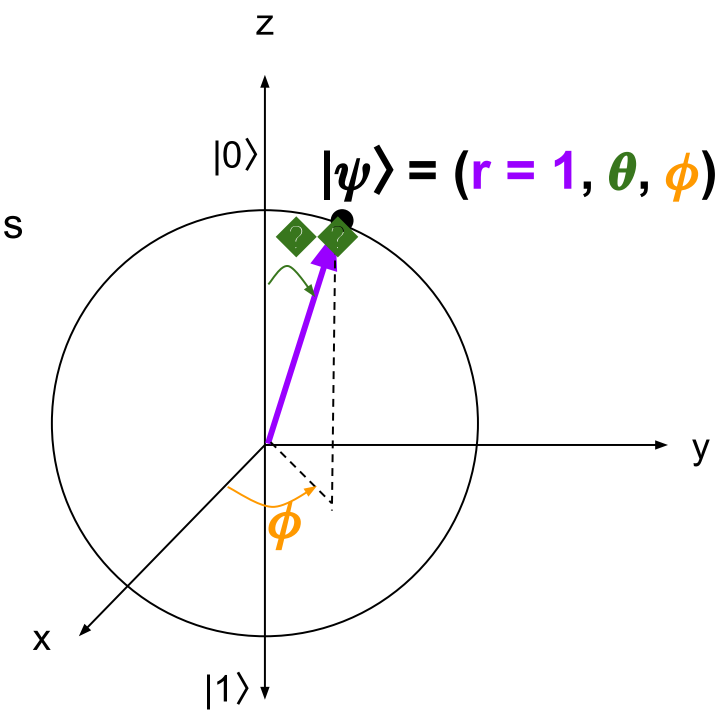
The H gate implements a rotation of π/2 around the y-axis of the Bloch Sphere!

The Bloch Sphere: A Deeper Look

- Qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ falls on sphere surface
- Requirement that $1 = |\alpha|^2 + |\beta|^2$ makes radius, **r = 1**
- Single qubit quantum operations implement rotations around x, y, and z axis to transform the state of $|\psi\rangle$
- New way to describe qubit:

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$

Probability amplitudes ***Phase***



Example: Confirm Location of $|-\rangle$ = $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$

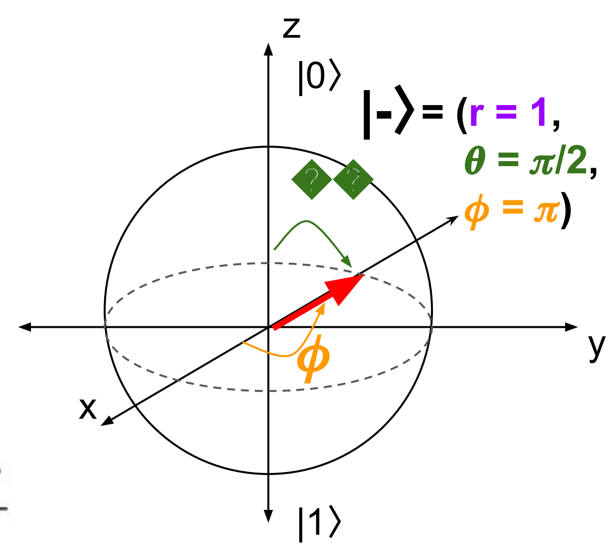
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\psi\rangle = \cos(\ (\pi/2)\ /2)|0\rangle + e^{i\pi}\sin(\ (\pi/2)\ /2)|1\rangle$

$|\psi\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

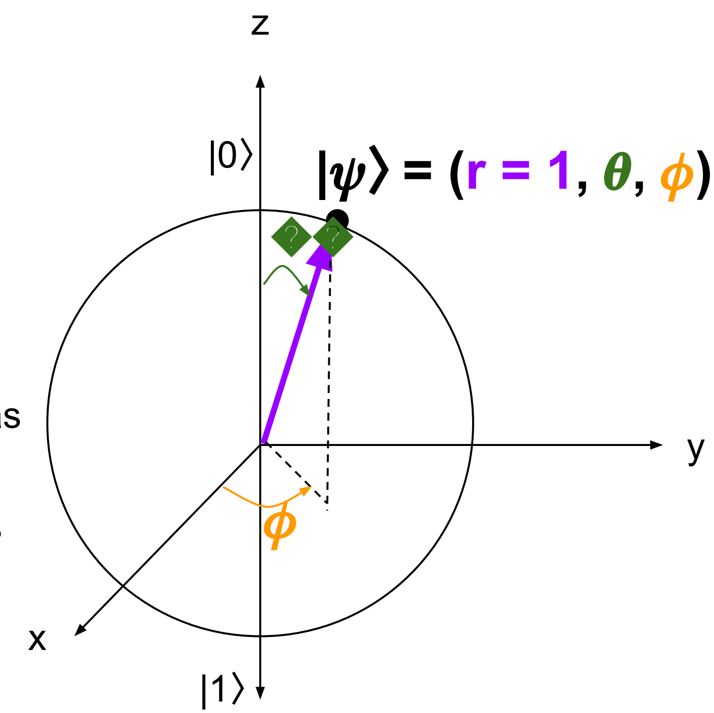
Note: $e^{ix} = \cos(x) + i^*\sin(x)$

f(x) \ x	**0**	***π/4***	***π/3***	***π/2***	***π***
cos(x)	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1
sin(x)	0	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0



Important Observations

- Angle θ influences the probability of observing $|0\rangle$ or $|1\rangle$
  - As θ increases, you are more likely to observe a $|1\rangle$
- Angle ϕ influences phase between $|0\rangle$ and $|1\rangle$
  - Phase never ***measured***, but important for quantum algorithms
  - We refer to the phase difference between $|0\rangle$ and $|1\rangle$ as ***relative phase*** (ex. the -1 in $|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$)
- ***Global phase*** common to both $|0\rangle$ and $|1\rangle$ expressed as $e^{i\gamma}|\psi\rangle$ (ex. $e^{i\gamma} = -1, i$) has no observable impact on quantum state and is typically discarded
  - $-|\psi\rangle = -(\alpha|0\rangle + \beta|1\rangle) = -\alpha|0\rangle - \beta|1\rangle \cong \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$



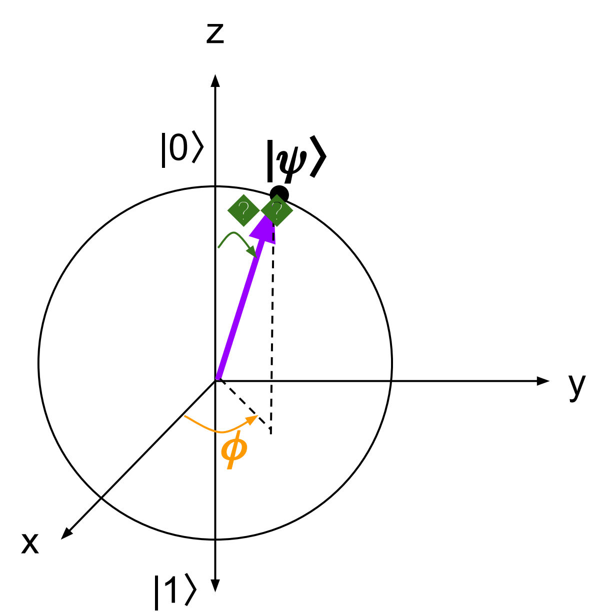
Generalized Rotation Operations

The generalized single-qubit rotation operations allow for arbitrary state transformation

$$R_x(\gamma) = \begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) & -i\sin\left(\frac{\gamma}{2}\right) \\ -i\sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) \end{bmatrix}$$

$$R_y(\gamma) = \begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) & -\sin\left(\frac{\gamma}{2}\right) \\ \sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) \end{bmatrix}$$

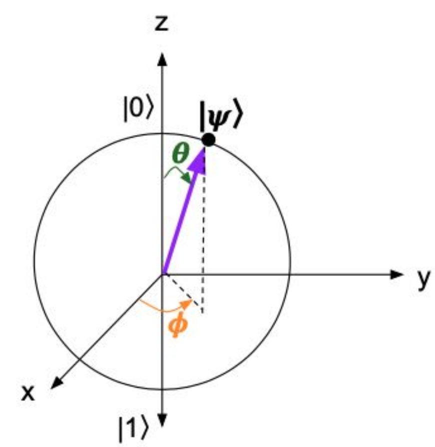
$$R_z(\gamma) = \begin{bmatrix} e^{-\frac{i\gamma}{2}} & 0 \\ 0 & e^{\frac{i\gamma}{2}} \end{bmatrix}$$



Example Rotation: Calculating $R_x(\pi)$

The $R_x(\pi)$ gate implements a rotation of π around the x-axis of the Bloch Sphere

$$R_x(\gamma) = \begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) & -i\sin\left(\frac{\gamma}{2}\right) \\ -i\sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) \end{bmatrix}$$



$$R_x(\pi) = \begin{bmatrix} \cos\left(\frac{\pi}{2}\right) & -i\sin\left(\frac{\pi}{2}\right) \\ -i\sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{bmatrix}$$

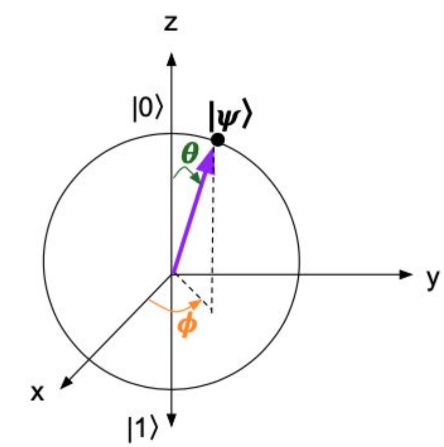
Note: $e^{ix} = \cos(x) + i^*\sin(x)$

f(x) \ x	**0**	**$\pi/4$**	**$\pi/3$**	**$\pi/2$**	**π**
cos(x)	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
sin(x)	0	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0

$$R_x(\pi) = -i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \mathbf{NOT} = \mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

*****Global phase (-i) has no observable impact on quantum state...we say that $R_x(\pi)$ is equal to X up to a global phase of -i***

Example Rotation: Calculating $R_z(\pi)$



The $R_z(\pi)$ gate implements a rotation of π around the z-axis of the Bloch Sphere

$$R_z(\gamma) = \begin{bmatrix} e^{-\frac{i\gamma}{2}} & 0 \\ 0 & e^{\frac{i\gamma}{2}} \end{bmatrix}$$

$$R_z(\pi) = \begin{bmatrix} e^{-\frac{i\pi}{2}} & 0 \\ 0 & e^{\frac{i\pi}{2}} \end{bmatrix}$$

$$R_z(\pi) = -i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

*****Global phase (-i) has no observable impact on quantum state...we say that $R_z(\pi)$ is equal to Z (phase flip) up to a global phase of -i***

Note: $e^{ix} = \cos(x) + i*\sin(x)$

f(x) \ x	**0**	**π/4**	**π/3**	**π/2**	**π**
cos(x)	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1
sin(x)	0	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0

PRACTICE:

What is the matrix for Ry(pi)?

$$R_y(\gamma) = \begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) & -\sin\left(\frac{\gamma}{2}\right) \\ \sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) \end{bmatrix}$$

a $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

c $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$

d $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Note: $e^{ix} = \cos(x) + i*\sin(x)$

f(x) \ x	**0**	***π*/4**	***π*/3**	***π*/2**	***π***
cos(x)	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1
sin(x)	0	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0

PRACTICE:

What is the matrix for Ry(pi)?

$$R_y(\gamma) = \begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) & -\sin\left(\frac{\gamma}{2}\right) \\ \sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) \end{bmatrix}$$

Note: $e^{ix} = \cos(x) + i*\sin(x)$

f(x) \ x	**0**	**π/4**	**π/3**	**π/2**	**π**
cos(x)	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1
sin(x)	0	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0

a $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

c $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$

d $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Gate is similar to Y gate (up to global phase) which is:

$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Takeaway

- Single qubit gates can be thought of as rotation operations
  - Visualized as rotations around the x-, y-, and z-axis of the Bloch Sphere
- The R_x, R_y, and R_z operations allow arbitrary qubit states to be created
  - Global phase on qubit state can generally be ignored