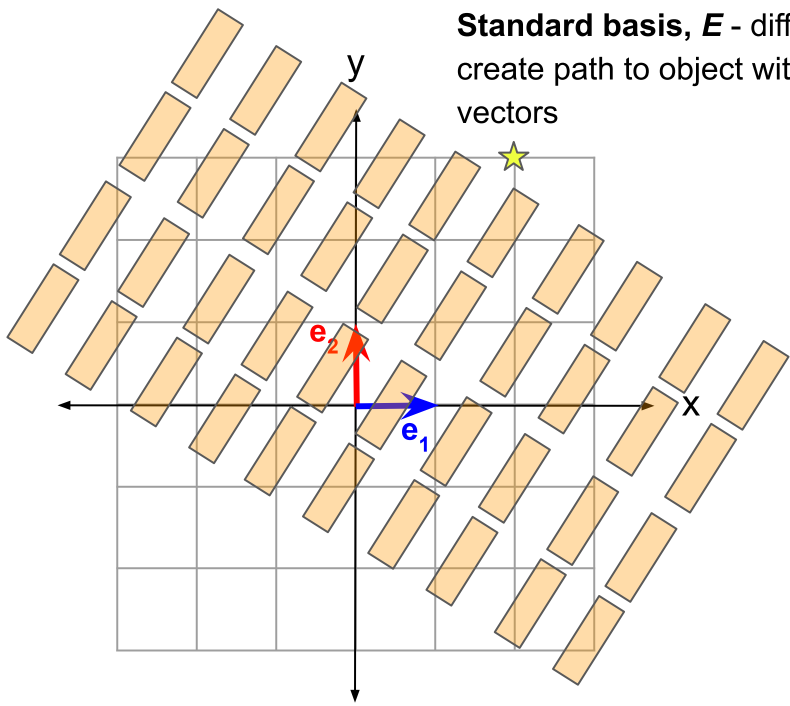
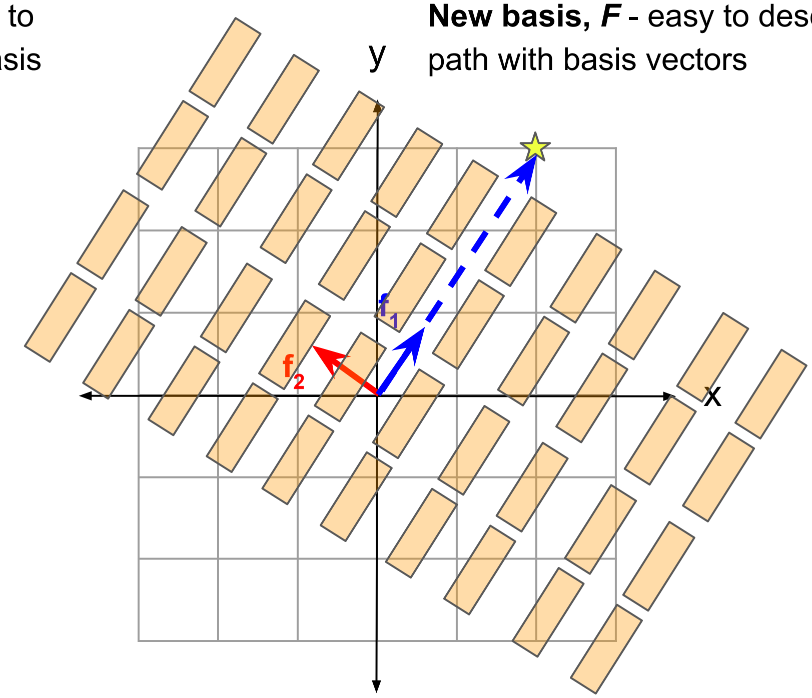
Change of Basis

Choosing a Reference for Locating an Object



Standard basis, *E* - difficult to create path to object with basis vectors



New basis, *F* - easy to describe path with basis vectors

A different reference (basis) can make state description more clear!

Review: Quantum State has a Vector Notation

$$a|0\rangle + b|1\rangle \quad \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \quad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \quad \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

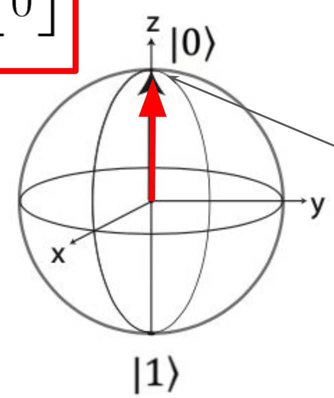
$$\frac{1}{2}|1\rangle + \frac{\sqrt{3}}{2}|0\rangle \quad \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

The Computational Basis

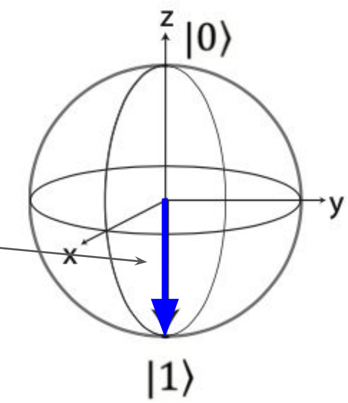
Typically, qubits are expressed using the ***computational basis*** corresponding to probability of measuring 0 or 1.

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$: Top of Bloch Sphere

$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$: Bottom of Bloch Sphere



The computational basis rests on the z-axis

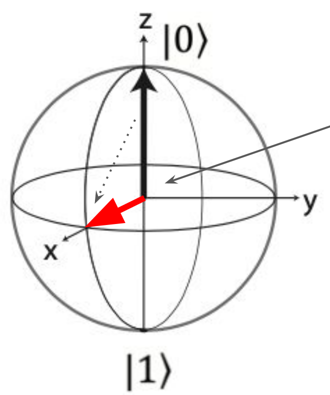


The Hadamard Basis

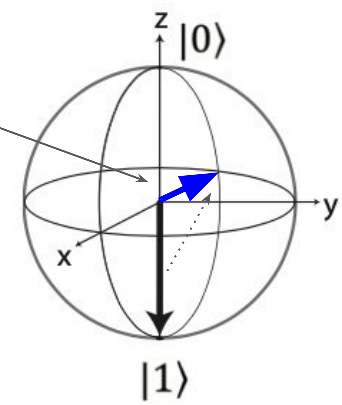
The ***Hadamard basis*** can also be used in QIS.

$$|+\rangle \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



The Hadamard, or +/- basis, rests on the x-axis



Another way to think about it….

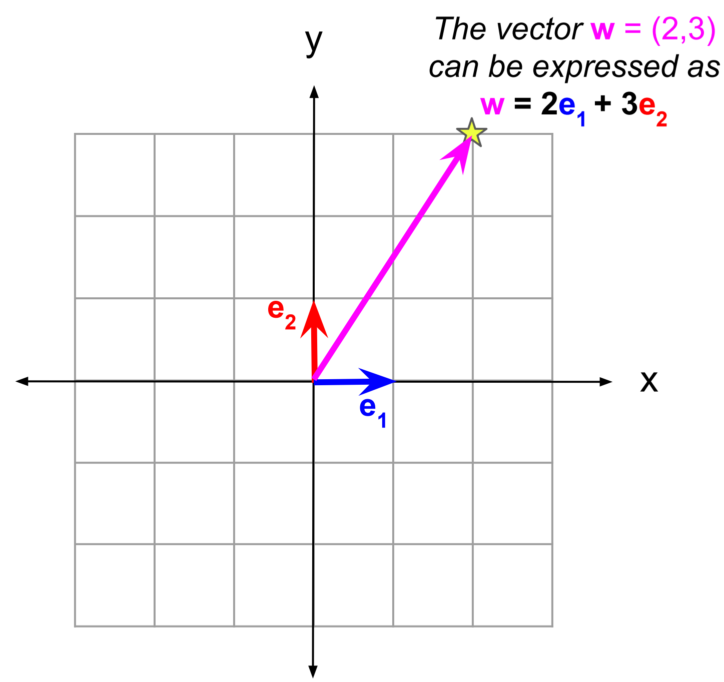
Conventional operations rotate the vector within the sphere

Basis changes rotate the sphere and vector together

Basis of a Vector Space

- A ***basis***, $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, is a set of vectors that can be used to express any point in a vector space, *V*, as a linear combination
- Different bases can be chosen for *V*, and choice of basis can make a problem easier to solve
- The standard basis, *E*, is used to describe vectors in the x-y plane:

$$E = \{\mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)\}$$



Change of Basis, Computational and Hadamard

- Linear transform with a matrix, *M,* allows base conversion
- $M_{H,C}$ to convert from the computational basis to the Hadamard basis
- $M_{C,H}$ to convert from the Hadamard basis to the computational basis

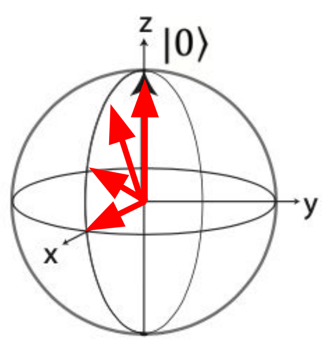
$$M_{H,C} = (M_{C,H})^{-1}$$

$$M_{H,C} = H = H^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Mapping Coordinates: Computational to Hadamard Basis

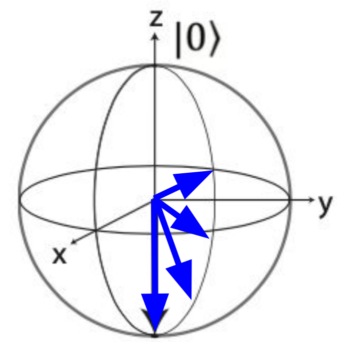
$$|0\rangle \Rightarrow |+\rangle$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\times\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

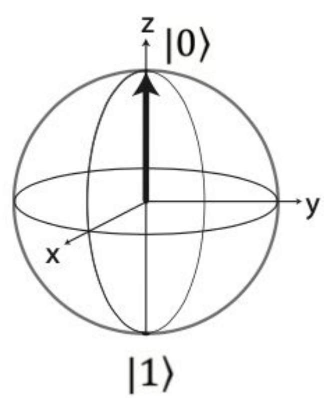


$$|1\rangle \Rightarrow |-\rangle$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\times\begin{bmatrix}0\\1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$



PRACTICE: True or False - Computational basis states $|0\rangle$ and $|1\rangle$ lie on opposite sides of the Bloch Sphere, but $|+\rangle$ and $|-\rangle$ do not.

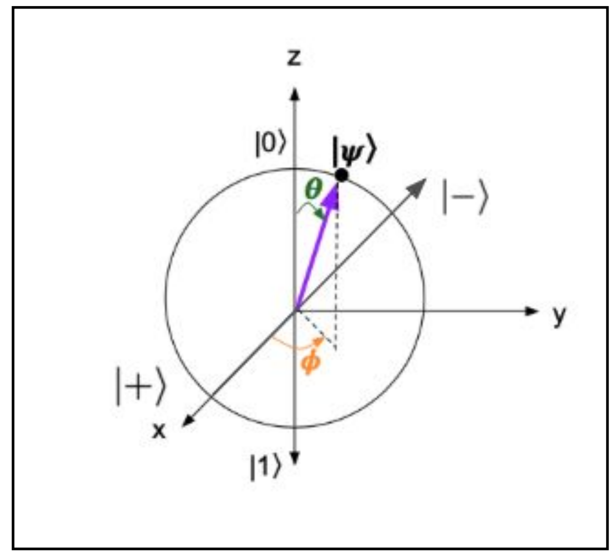


a. True
b. False
c. Sometimes true
d. I don’t know

PRACTICE: True or False - Computational basis states $|0\rangle$ and $|1\rangle$ lie on opposite sides of the Bloch Sphere, but $|+\rangle$ and $|-\rangle$ do not.

B. FALSE. $|+\rangle$ and $|-\rangle$ are also on opposite sides, just on the x-axis!

Using vectors located on opposite sides of the Bloch sphere is actually important if you are trying to define a basis for your quantum information!



Example 1: Basis Change of a Qubit at State $|0\rangle$

A qubit has a state of $|v\rangle = |0\rangle$ Express the qubit as $|v\rangle = v_+|+\rangle + v_-|-\rangle$.

Known: $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Solution:

$$|v\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} \end{bmatrix}$$

Example 1: Basis Change of a Qubit, Generalized

A qubit has a state $|v\rangle = v_0|0\rangle + v_1|1\rangle$ xpress the qubit as $|v\rangle = v_+|+\rangle + v_-|-\rangle$

Known: $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Solution: $$|v\rangle = \frac{v_0 + v_1}{\sqrt{2}}|+\rangle + \frac{v_0 - v_1}{\sqrt{2}}|-\rangle$$

Digging Deeper
Rules for Qubit Basis Sets

A set of basis vectors, $X = \{\mathbf{x}_1, \mathbf{x}_2\}$, must be ***orthonormal***:

- All included vectors are orthogonal (perpendicular)

$$\langle \mathbf{x}_1 | \mathbf{x}_2 \rangle = 0$$

- Each vector has a norm (distance) of 1

$$\langle \mathbf{x}_1 | \mathbf{x}_1 \rangle = |\mathbf{x}_1| = 1$$

Vector Transpose:

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \alpha_0 & \alpha_1 \end{bmatrix}$$

Complex Conjugate:

$$c = a + bi$$
$$\bar{c} = a - bi$$

Bra-Ket Notation: |Ket⟩to⟨Bra| conversion

$$\langle \boldsymbol{z}| = |\boldsymbol{z}\rangle^{\dagger} = |\bar{\boldsymbol{z}}\rangle^{T}$$

Euclidean Distance/Norm:

$$|\boldsymbol{z}| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}$$
$$= \sqrt{z_1\bar{z_1} + z_2\bar{z_2} + \cdots + z_n\bar{z_n}}$$

Advanced Mathematics For QIS

Qubit state and quantum gates are represented by vectors/matrices

- Elements can be complex!

Special linear algebra operations are needed to analyze quantum states and state transformations with complex values. Some important operations include:

- ***Complex conjugate*** - Helps calculate state probabilities with complex values
- ***Conjugate transpose -*** Allows qubit state column vectors (|Kets〉) to be transformed to row vectors (〈Bras|)
- ***Inner Product*** - Applied to confirm that basis states (|0〉/|1〉, |+〉/|-〉, |i〉/|-i〉) can be used to describe the state of a qubit

Complex Conjugate

The complex conjugate operation changes the value of the imaginary component of a complex number

$$\text{If } z = a + bi,$$
$$\text{then}$$

Complex Conjugate

$$\bar{z} = a - bi$$

"Hat" indicates conjugate operation

'Flips' the sign of imaginary values!

Note: If a number is not complex, it is equal to its complex conjugate

$z = a + 0*i = \bar{z} = a - 0*i$

Example: Complex Conjugate

If $z = a + bi$, *then*

Complex Conjugate

$$\bar{z} = a - bi$$

$z = 8 + 3i$

$\bar{z} = 8 - 3i$

$c = 12 - 7i$

$\bar{c} = 12 + 7i$

$v = (1 + i)/\sqrt{2}$

$\bar{v} = (1 - i)/\sqrt{2}$

Key Linear Algebra Operation: Transpose

Transpose operations are used on quantum states and operations to adjust the row and column values

Vector Transpose:

$$\begin{bmatrix}\alpha_0 \\ \alpha_1\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}\alpha_0 & \alpha_1\end{bmatrix} , \begin{bmatrix}\alpha_0 & \alpha_1 \\ \alpha_2 & \alpha_3\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}\alpha_0 & \alpha_2 \\ \alpha_1 & \alpha_3\end{bmatrix}$$

"T" indicates transpose

'Flips' a vector/matrix around a diagonal

Example: Transpose

Vector Transpose:

$$\begin{bmatrix}\alpha_0 \\ \alpha_1\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}\alpha_0 & \alpha_1\end{bmatrix}, \quad \begin{bmatrix}\alpha_0 & \alpha_1 \\ \alpha_2 & \alpha_3\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}\alpha_0 & \alpha_2 \\ \alpha_1 & \alpha_3\end{bmatrix}$$

$$M = \begin{bmatrix}-2 & 5 \\ 1 & 9\end{bmatrix}$$

$$M^T = \begin{bmatrix}-2 & 1 \\ 5 & 9\end{bmatrix}$$

$$U = \begin{bmatrix}1 & 2 & 3 \\ 4 & 5 & 6\end{bmatrix}$$

$$U^T = \begin{bmatrix}1 & 4 \\ 2 & 5 \\ 3 & 6\end{bmatrix}$$

$$v = \begin{bmatrix}1 \\ 0 \\ 0 \\ 0\end{bmatrix}$$

$$v^T = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}$$

PRACTICE: What is the transpose of the following matrix?

$$\begin{pmatrix} -2 & 9 & 0 \\ 3 & 6 & 8 \end{pmatrix}$$

a. $\begin{pmatrix} -2i & 9i & 0 \\ 3i & 6i & 8i \end{pmatrix}$

b. $\begin{pmatrix} -2i & 3i \\ 9i & 6i \\ 0 & 8 \end{pmatrix}$

c. $\begin{pmatrix} -2 & 3 \\ 9 & 6 \\ 0 & 8 \end{pmatrix}$

d. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

PRACTICE: What is the transpose of the following matrix?

$$\begin{pmatrix} -2 & 9 & 0 \\ 3 & 6 & 8 \end{pmatrix}$$

a. $\begin{pmatrix} -2i & 9i & 0 \\ 3i & 6i & 8i \end{pmatrix}$

b. $\begin{pmatrix} -2i & 3i \\ 9i & 6i \\ 0 & 8 \end{pmatrix}$

c. $\begin{pmatrix} -2 & 3 \\ 9 & 6 \\ 0 & 8 \end{pmatrix}$

d. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Conjugate Transpose of a Vector/Matrix

Invert the sign of all *imaginary* values (***complex conjugate***), and mirror vector or matrix elements (***transpose***) across the diagonal!

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}^\dagger = \begin{bmatrix} \bar{\alpha_0} & \bar{\alpha_1} \end{bmatrix}, \quad \begin{bmatrix} \alpha_0 & \alpha_1 \\ \alpha_2 & \alpha_3 \end{bmatrix}^\dagger = \begin{bmatrix} \bar{\alpha_0} & \bar{\alpha_2} \\ \bar{\alpha_1} & \bar{\alpha_3} \end{bmatrix}$$

$$\begin{bmatrix} a+ib & c+id \\ e+if & g+ik \end{bmatrix}^\dagger = \begin{bmatrix} a-ib & e-if \\ c-id & g-ik \end{bmatrix}$$

"Dagger" indicates conjugate transpose

Example:
Conjugate Transpose

Conjugate Transpose

$$\begin{bmatrix}\alpha_0 \\ \alpha_1\end{bmatrix}^\dagger = \begin{bmatrix}\bar{\alpha_0} & \bar{\alpha_1}\end{bmatrix}, \quad \begin{bmatrix}\alpha_0 & \alpha_1 \\ \alpha_2 & \alpha_3\end{bmatrix}^\dagger = \begin{bmatrix}\bar{\alpha_0} & \bar{\alpha_2} \\ \bar{\alpha_1} & \bar{\alpha_3}\end{bmatrix}$$

$$\begin{bmatrix}a+ib & c+id \\ e+if & g+ik\end{bmatrix}^\dagger = \begin{bmatrix}a-ib & e-if \\ c-id & g-ik\end{bmatrix}$$

$$v = \begin{bmatrix}1-i \\ 0 \\ 3+4i \\ 0\end{bmatrix}$$

$$v^\dagger = \begin{bmatrix}1+i & 0 & 3-4i & 0\end{bmatrix}$$

$$M = \begin{bmatrix}3i & 1+i \\ 1 & -5i\end{bmatrix}$$

$$M^\dagger = \begin{bmatrix}-3i & 1 \\ 1-i & 5i\end{bmatrix}$$

PRACTICE: What is the conjugate transpose of the following matrix?

$$\begin{pmatrix} -2 & 9i & i \\ -3i & 6 & 8 \end{pmatrix}$$

a. $\begin{pmatrix} 2i & -9i & -i \\ 3i & 6 & 8 \end{pmatrix}$

b. $\begin{pmatrix} -2 & 3i \\ -9i & 6 \\ -i & 8 \end{pmatrix}$

c. $\begin{pmatrix} -2 & 3i \\ 9 & 6 \\ i & 8 \end{pmatrix}$

d. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

PRACTICE: What is the conjugate transpose of the following matrix?

$$\begin{pmatrix} -2 & 9i & i \\ -3i & 6 & 8 \end{pmatrix}$$

a. $\begin{pmatrix} 2i & -9i & -i \\ 3i & 6 & 8 \end{pmatrix}$

b. $\begin{pmatrix} -2 & 3i \\ -9i & 6 \\ -i & 8 \end{pmatrix}$

c. $\begin{pmatrix} -2 & 3i \\ 9 & 6 \\ i & 8 \end{pmatrix}$

d. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

|Ket⟩to⟨Bra| conversion

Quantum analysis requires |Kets⟩to be transformed to⟨Bras| (and vice versa!)

- Both transformations use the same steps
- Conjugate transpose: complex conjugate then vector transpose

$$\langle \mathbf{z}| = |\mathbf{z}\rangle^{\dagger} = |\bar{\mathbf{z}}\rangle^{T}$$

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}^{\dagger} = \begin{bmatrix} \bar{\alpha}_0 & \bar{\alpha}_1 \end{bmatrix}$$

Example: |Ket⟩to⟨Bra|

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$

$$|0\rangle^{\dagger} = \langle 0| = \begin{bmatrix}1 & 0\end{bmatrix}$$

$$|\phi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$$

$$|\phi\rangle^{\dagger} = \langle \phi| = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\end{bmatrix}$$

$$|\psi\rangle = \begin{bmatrix}0\\-i\end{bmatrix}$$

$$|\psi\rangle^{\dagger} = \langle \psi| = \begin{bmatrix}0 & i\end{bmatrix}$$

Note:⟨Bra| to |Ket⟩conversion follows the same procedure!

$$\langle \psi|^{\dagger} = |\psi\rangle$$

PRACTICE: Convert $|10\rangle$ to $\langle 10|$

$$\langle \mathbf{z}| = |\mathbf{z}\rangle^\dagger = |\overline{\mathbf{z}}\rangle^T$$

A. $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

B. $\begin{pmatrix} 0 \\ 0 \\ i \\ 0 \end{pmatrix}$

C. $\begin{pmatrix} 0 & 0 & i & 0 \end{pmatrix}$

D. $\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$

PRACTICE: Convert $|10\rangle$ to $\langle 10|$

$$\langle \mathbf{z}| = |\mathbf{z}\rangle^{\dagger} = |\bar{\mathbf{z}}\rangle^{T}$$

A. $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

B.

C. $\begin{pmatrix} 0 \\ 0 \\ i \\ 0 \end{pmatrix}$

D. $\begin{pmatrix} 0 & 0 & i & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$

Inner Product

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$

$$|\phi\rangle = \beta_0|0\rangle + \beta_1|1\rangle$$

The ***inner product*** of two quantum states $|\psi\rangle$ and $|\phi\rangle$ is defined as

$$\langle\psi| \cdot |\phi\rangle = \langle\psi|\,|\phi\rangle$$

Must convert from Ket to Bra

$$\langle\psi|\phi\rangle = \begin{bmatrix}\overline{\alpha_0} & \overline{\alpha_1}\end{bmatrix}\begin{bmatrix}\beta_0 \\ \beta_1\end{bmatrix}$$

$$= \overline{\alpha_0}\beta_0 + \overline{\alpha_1}\beta_1$$

The result of an inner product is a scalar (single value)

Example: Inner Product

$$\langle +|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} * 1 + \frac{1}{\sqrt{2}} * 0$$

$$= \frac{1}{\sqrt{2}}$$

Digging Deeper
Rules for Qubit Basis Sets

A set of basis vectors, $X = \{\mathbf{x}_1, \mathbf{x}_2\}$, must be ***orthonormal***:

- All included vectors are orthogonal (perpendicular)

$$\langle \mathbf{x}_1 | \mathbf{x}_2 \rangle = 0$$

- Each vector has a norm (distance) of 1

$$\langle \mathbf{x}_1 | \mathbf{x}_1 \rangle = |\mathbf{x}_1| = 1$$

Vector Transpose:

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \alpha_0 & \alpha_1 \end{bmatrix}$$

Complex Conjugate:

$$c = a + bi$$
$$\bar{c} = a - bi$$

Bra-Ket Notation: |Ket⟩to⟨Bra| conversion

$$\langle \boldsymbol{z} | = | \boldsymbol{z} \rangle^{\dagger} = | \bar{\boldsymbol{z}} \rangle^{T}$$

Euclidean Distance/Norm:

$$|\boldsymbol{z}| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2} = \sqrt{z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n}$$

Vector Distance

Euclidean distance (also called ***Euclidean norm***) is the distance of a vector from 'tip to tail'

$$\begin{aligned} |\mathbf{z}| &= \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2} \\ &= \sqrt{z_1\overline{z_1} + z_2\overline{z_2} + \cdots + z_n\overline{z_n}} \end{aligned}$$

Applied Complex Linear Algebra: Rules for Qubit Basis Sets

A set of basis vectors, $X = \{\mathbf{x}_1, \mathbf{x}_2\}$, must be ***orthonormal***:

- All included vectors are orthogonal (perpendicular)

$$\langle \mathbf{x}_1 | \mathbf{x}_2 \rangle = 0$$

- Each vector has a norm (distance) of 1

$$\langle \mathbf{x}_1 | \mathbf{x}_1 \rangle = |\mathbf{x}_1| = 1$$

Vector Transpose:

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Euclidean Distance/Norm:

$$|\boldsymbol{z}| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2} = \sqrt{z_1\overline{z_1} + z_2\overline{z_2} + \cdots + z_n\overline{z_n}}$$

Example: Check Orthonormality

The Computational Basis: $|0\rangle$ and $|1\rangle$

- All included vectors are orthogonal (perpendicular) , $\langle \mathbf{x_1} | \mathbf{x_2} \rangle = 0$

$$\langle 0|1\rangle = \begin{bmatrix}1\\0\end{bmatrix}^{\dagger} \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 0$$

- Each vector has a norm (distance) of 1, $\langle \mathbf{x_1} | \mathbf{x_1} \rangle = |\mathbf{x_1}| = 1$

$$\langle 0|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}^{\dagger} \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 1$$

$$\langle 1|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}^{\dagger} \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 1$$