CS 221/321	**Programming Languages**	**Homework 2**
Fall 2009		**Due: Oct 20, 2009**

1. [10 points] Free and bound variables and substitution in the λ-calculus.

We can enrich the pure lambda calculus a bit by adding constants for natural numbers and primitive arithmetic operations like $+$ and $-$, allowing terms like $\lambda x.x + 2$. The expressions of this enriched calculus are defined by the following grammar:

$$M ::= n \mid x \mid M_1 + M_2 \mid M_1 - M_2 \mid M_1\, M_2 \mid \lambda x.M$$

where n ranges over natural number constants and x ranges over a set of variables. The juxtaposition of two expressions, $M_1 M_2$, denotes the application of a function (M_1) to an argument (M_2), and function application associates to the left, so that $M_1 M_2 M_3$ is read as $(M_1 M_2)M_3$. In a lambda abstraction $\lambda x.M$, the variable x is bound in its scope M. In this enriched calculus a *value* is either a lambda abstraction or an integer constant.

(1.a) [5 Points] For the lambda terms M_0 and M_1 below, what variables appear free in M_0, and what variables appear bound? Mark bound variable uses with an underline, and free variables with an overline, and draw a line connecting each use of a bound variable with the corresponding binding occurrence of that variable. Pay careful attention to the structure of the expression as determined by the parentheses.

$$M_0 = (\lambda g \,.\, g\ (f\ 2))\ ((\lambda y \,.\, \lambda z \,.\, \lambda x \,.\, z\ (\lambda u \,.\, u + x))\ x\ (\lambda f \,.\, f\ y))$$

$$M_1 = \lambda y \,.\, x\ (\lambda z \,.\, \lambda x \,.\, y(\lambda z \,.\, z\ x))\ z$$

(1.b) [5 points] Calculate the results of the capture-free substitutions $\{g/f\}M_0$ and $\{f/g\}M_0$.

2. [20 points] Do exercise 1 of Section 6.2 (page 38). Give a precise statement of any induction hypothesis used in the inductive proofs.

3. [20 points] Do exercise 1 of Section 7.4 (page 44). Give a precise statement of any induction hypothesis used in the inductive proof.