CS 153	**Foundations of Software**	**Homework Solution 7**
Fall 2006		**Due November 20, 2006**

1. [5] Exercise 7.1.1(b) (p. 413) [Hint: Use answer to part (a) as a model of what is expected.]
Solution:
$[p(0,0) \vee p(1,0)] \wedge [p(0,1) \vee p(1,1)]$

2. [5] Exercise 7.1.5(b) (p. 413)
Solution:
The parenthesized form of $\forall y\ q(y) \wedge \neg p(x,y)$ is $(\forall y\ q(y)) \wedge (\neg p(x,y))$. Therefore, the only occurrence of x is free, the first two occurrences of y are bound, and the third occurrence of y is free.

3. [5] Exercise 7.1.7(b) (p. 414)
Solution:
Anything that eats is a bird, and anything that is eaten is a worm.

4. [5] Exercise 7.1.9(b) (p. 415)
Solution:
$\forall x\ \forall y\ (\neg e(x,a) \wedge \neg e(y,a) \rightarrow \exists z\ (d(z,x) \wedge d(z,y)))$, where $a = 0$.

5. [10] Exercise 7.1.10(b) (p. 415)
Solution:
$W = \exists x\ p(x) \rightarrow \forall x\ p(x)$
Case $p(a)$ = true and $p(b)$ = true: $\exists x\ p(x)$ is true and $\forall x\ p(x)$ is true. Therefore, W is true.
Case $p(a)$ = true and $p(b)$ = false: $\exists x\ p(x)$ is true and $\forall x\ p(x)$ is false. Therefore, W is false.
Case $p(a)$ = false and $p(b)$ = true: $\exists x\ p(x)$ is true and $\forall x\ p(x)$ is false. Therefore, W is false.
Case $p(a)$ = false and $p(b)$ = false: $\exists x\ p(x)$ is false and $\forall x\ p(x)$ is false. Therefore, W is true.

6. [10] Exercise 7.1.11(b) (p. 415)
Solution:
Let the domain be the set $\{a, b\}$, and assign $p(a)$ = true and $p(b)$ = true. We have that $\exists x\ p(x)$ is true and $\forall x\ p(x)$ is true. Therefore, $\exists x\ p(x) \rightarrow \forall x\ p(x)$ is true. Another example would be a domain $\{a\}$ such that $p(a)$ is true.

7. [10] Exercise 7.1.12(d) (p. 415)
Solution:
Let the domain be the set $\{a, b\}$, and assign $p(a)$ = true, $p(b)$ = false, $q(a)$ = false, and $q(b)$ = true. $\exists x\ p(x)$ is true, because $p(a)$ = true. $\exists x\ q(x)$ is true, because $q(b)$ = true. Therefore, $\exists x\ p(x) \wedge \exists x\ q(x)$ is true. $p(a) \wedge q(a)$ and $p(b) \wedge q(b)$ are false. Therefore, $\exists x\ (p(x) \wedge q(x))$ is

false. So we have that $\exists x\ p(x) \wedge \exists x\ q(x) \rightarrow \exists x\ (p(x) \wedge q(x))$ is false.

8. [10] Exercise 7.1.14(b) (p. 415)
Solution:
Let the domain be the set $\{a, b\}$. First, we assume that $p(a, a)$ is false or $p(b, b)$ is false. $\forall x\ p(x, x)$ is false. Therefore, W is true. Next, we assume that both $p(a, a)$ and $p(b, b)$ are true. Because x,y, and z take values in $\{a, b\}$, for any assignment x,y, and z, we can find two distinct variables $r, s \in \{x, y, z\}$ such that $r = s = a$ or $r = s = b$, and in either case $p(r, s)$ is true. So $p(x, y) \vee p(x, z) \vee p(y, z)$ is true for every assignment x,y, and z. That is, $\forall x\ \forall y\ \forall z\ (p(x, y) \vee p(x, z) \vee p(y, z))$ is true, and hence W is true. In conclusion, W is true for any interpretation.

9. [5] Exercise 7.1.16(b) (p. 416)
Solution:
Let $W = p(c) \rightarrow \exists x\ p(x)$. If $p(c) =$ true, $\exists x\ p(x)$ is true. Therefore, W is true. If $p(c) =$ false, W is true. Hence W is valid.

10. [5] Exercise 7.1.17(b) (p. 415)
Solution:
We prove it by contradiction. If $\exists x\ (p(x) \wedge \neg p(x))$ is satisfiable, there is a c in the domain satisfying $p(c) \wedge \neg p(c)$. But this formula is a contradiction (always false).

11. [30] Write (and test and debug) a program in Standard ML (SML) that represents terms and formulas of a predicate calculus over a language that has the function symbols e (nullary) and $*$ (binary) and an relation symbol *eq* (these function symbols might represent the identity element and multiplication operator of a *monoid* – look it up in Wikipedia if you are not familiar with monoids). You can use the supplied file pc.sml as a starting place (see the class web page).

(a) Write a function that calculates the set of free variables of a formula. Then use this to write functions that produce the universal and existential closures of a given formula.

(b) Write a function that implements substitution of a term for free occurrences of a variable in a formula (`subst(A,x,t)`$=A[t/x]$). What can you do about situations like the following? $(\exists x\, p(x, y))[f(x)/y] =_? \exists x\, p(x, f(x))$ (an instance of undesirable *free variable capture*).