CS 153	**Foundations of Software**	**Homework Solution 6**
Fall 2006		**Due Nov 13, 2006**

1. [5] Exercise 6.3.3(b) (p. 382)

(b) $((A \to B) \to C) \to D$

The formula $(A \to B) \to C$ is the single premiss for a proof of D by Conditional Proof, or equivalently, the $\to$ intro rule. Here is the last inference rule of a natural deduction sequent style proof of the formula.

$$\frac{(A \to B) \to C \vdash D}{\vdash ((A \to B) \to C) \to D} \ (\to \text{intro})$$

2. [15] Exercise 6.3.5(b,d) (p. 382)

Proof the following using the CP rule:

(b) $A \to (\neg B \to (A \wedge \neg B))$

1.	A	P
2.	$\neg B$	P
3.	$A \wedge \neg B$	1, 2, Conj
4.	$\neg B \to (A \wedge \neg B)$	2, 3, CP
5.	$A \to (\neg B \to (A \wedge \neg B))$	1, 4, CP

(d) $(B \to C) \to (A \wedge B \to A \wedge C)$

1.	$B \to C$	P
2.	$A \wedge B$	P
3.	A	2, Simp
4.	B	2, Simp
5.	C	1, 4, MP
6.	$A \wedge C$	3, 5, Conj
7.	$(A \wedge B \to A \wedge C)$	2, 6, CP
8.	$(B \to C) \to (A \wedge B \to A \wedge C)$	1, 7, CP

3. [15] Exercise 6.3.6(d,f) (p. 382)

Give formal proofs for the following tautologies using the IP rule:

(d) $G = (A \to B) \to (C \vee A \to C \vee B)$
We assume $\neg G$, which is equivalent to the conjunction of $A \to B$, $C \vee A$, and $\neg(C \vee B)$.

1.	$(A \to B)$	P
2.	$C \vee A$	P
3.	$\neg(C \vee B)$	P
4.	$\neg C \wedge \neg B$	3, Taut
5.	$\neg C$	4, Simp
6.	$\neg B$	4, Simp
7.	A	2,5, DS
8.	B	1,7, MP
9.	$B \wedge \neg B$	6,8,Conj
10.	F	9, Taut
11.	G	1,2,3,10, IP

(f) $G = (A \to B) \to ((B \to C) \to (A \vee B \to C)$ We assume $\neg G$, which is equivalent to the conjunction of $A \to B$, $B \to C$, $A \vee B$ and $\neg C$.

1.	$A \to B$	P
2.	$B \to C$	P
3.	$A \vee B$	P
4.	$\neg C$	P
5.	$\neg B$	2,4,MT
6.	A	3,5,DS
7.	B	1,6,MP
8.	$B \wedge \neg B$	5,7,Conj
9.	F	8,Taut
10.	G	1,2,3,4,9 IP

4. [10] Exercise 6.4.2 (p. 392)

(d) In Frege's axiom system, prove that $(\neg A \to \neg B) \to (B \to A)$.

As noted in Section 6.4.2, the CP rule (or Deduction Theorem) holds for Frege's axiom system, so we can employ the CP rule along with the axioms and Modus Ponens.

1.	$\neg A \to \neg B$	P
2.	B	P
2.	$(\neg A \to \neg B) \to (\neg\neg B \to \neg\neg A)$	Axiom 4
3.	$(\neg\neg B \to \neg\neg A)$	1, 3, MP
4.	$B \to \neg\neg B$	Axiom 6
5.	$\neg\neg B$	2,5,MP
6.	$\neg\neg A$	3,5,MP
7.	$\neg\neg A \to A$	Axiom 5
8.	A	6,7,MP
9.	$B \to A$	2,8,CP
10.	$(\neg A \to \neg B) \to (B \to A)$	1,9,CP

5. [20] Exercise 6.4.3(d,h) (p. 392)

In Hilbert-Ackermann axiom system prove the following

(d) $A \vee \neg A$
Note that we can use earlier parts of the exercise, in this case part (b).

1.	$A \to A \vee A$	Axiom 2
2.	$A \vee A \to A$	Axiom 1
3.	$A \to A$	1, 2, HS (Part (b))
4.	$\neg A \vee A$	3, Definiton of $\to$
5.	$\neg A \vee A \to A \vee \neg A$	4, Axiom 3
6.	$A \vee \neg A$	4, 5, MP

(h) $(A \to B) \to (\neg B \to \neg A)$
In this case, we use parts (b) and (e) of the exercise.

1.	$B \to \neg\neg B$	(Part (e) of Exercise 6.4.3)
2.	$(B \to \neg\neg B) \to ((\neg A \vee B) \to (\neg A \vee \neg\neg B))$	Axiom 4
3.	$(\neg A \vee B) \to (\neg A \vee \neg\neg B)$	1, 2, MP
4.	$(\neg A \vee \neg\neg B) \to (\neg\neg B \vee \neg A)$	Axiom 3
5.	$(\neg A \vee B) \to (\neg\neg B \vee \neg A)$	3, 4, HS (Part (b))
6.	$(A \to B) \to (\neg B \to \neg A)$	5, Definiton of $\to$