CS 153	**Foundations of Software**	**Homework Solution 3**
Fall 2006		**Due October 20, 2006**

1. [5] Express the properties of a partial order relation R being antisymmetric and total in terms R and its inverse R^{-1}.
Solution: Assume R is a partial order on a set A, so $R \subseteq A \times A$, and let I be the identity relation on A.

antisymmetric: If R is antisymmetric, then $\forall a,b \in A. (a,b) \in R \wedge (a,b) \in R^{-1} \Rightarrow a = b$, which implies $R \cap R^{-1} \subseteq I$. Conversely, if $R \cap R^{-1} \subseteq I$, then $(a,b) \in R$ and $(b,a) \in R$ imply that $(a,b) \in R \cap R^{-1}$, which implies that $(a,b) \in I$, or $a = b$, so R is antisymmetric. Thus R is antisymmetric iff $R \cap R^{-1} \subseteq I$.

total: If R is total, then for any $a,b \in A$, $(a,b) \in R$ or $(a,b) \in R^{-1}$. Therefore, $R \cup R^{-1} = A \times A$. Conversely, if $R \cup R^{-1} = A \times A$, then for any $a,b \in A$, $(a,b) \in R$ or $(a,b) \in R^{-1}$ (equivalently $(b,a) \in R$), so R is total. Thus R is total iff $R \cup R^{-1} = A \times A$.

2. Exercise 4.3.14 (b,d,f,h) (p. 253) [20 points]
Solution:
(b) f is monotonic. If $a,b \in \mathbb{N}$ and $a < b$, $f(a) = a^2 < b^2 = f(b)$
(d) f is not monotonic. A counter example: $1 \mid 2$, but $f(1) = 5 \nmid 7 = f(2)$. But $5 \nmid 7$.
(f) f is not monotonic. A counter example: $2 \mid 6$, but $f(2) = 2 \nmid 1 = f(6)$.
(h) f is monotonic. Let $A,B \in \mathscr{P}(\mathbb{N})$ and $A \subset B$. Suppose $k \in f(A)$. Then $k \mid a$ for some $a \in A$. Because $A \subset B$, we also have $a \in B$, so $k \in f(B)$. Therefore, $f(A) \subset f(B)$.

3. [10] Show that the composition of two monotonic functions between posets is a monotonic function.
Solution:
Let $\langle A, \preceq_A \rangle$,$\langle B, \preceq_B \rangle$, and $\langle C, \preceq_C \rangle$ be posets. Let $f : A \to B$ and $g : B \to C$ be monotonic functions. Let $x,y \in A$ with $x \preceq_A y$. Then $f(x) \preceq_B f(y)$ because f is monotonic, and thus $g(f(x)) \preceq_C g(f(y))$ because g is monotonic. Therefore, we have $(g \circ f)(x) = g(f(x)) \preceq_C g(f(y)) = (g \circ f)(y)$, i.e., $g \circ f$, the composition of f and g, is monotonic.

4. [5] Show that for any function $f : A \to B$, the associated image function $f : \mathscr{P}(A) \to \mathscr{P}(B)$ and the inverse image function $f^{-1} : \mathscr{P}(B) \to \mathscr{P}(A)$ are monotonic relative to the subset ordering on $\mathscr{P}(A)$ and $\mathscr{P}(B)$.
Solution:
Let $X_1, X_2 \in \mathscr{P}(A)$ and $X_1 \subset X_2$. For any $y \in B$, if $y \in f(X_1)$, there is an $x \in X_1$ such that $f(x) = y$. Because $X_1 \subset X_2$, $x \in X_2$. Therefore, $y = f(x) \in f(X_2)$. So we have $f(X_1) \subset f(X_2)$. Therefore, $f : \mathscr{P}(A) \to \mathscr{P}(B)$ is monotonic.
Let $Y_1, Y_2 \in \mathscr{P}(B)$ and $Y_1 \subset Y_2$. For any $x \in A$, if $x \in f^{-1}(Y_1)$, $f(x) = y \in Y_1$. Because $Y_1 \subset Y_2$, $y \in Y_2$. Therefore, $x \in f^{-1}(Y_2)$. So we have $f^{-1}(Y_1) \subset f^{-1}(Y_2)$. Therefore, $f^{-1} : \mathscr{P}(B) \to \mathscr{P}(A)$ is monotonic.

5. [15] Consider the set $\mathscr{R}$ of binary relations over a set A:

$$\mathscr{R} = \mathscr{P}(A \times A)$$

and let these relations be ordered by subset, so we are considering the poset $\langle \mathscr{R}, \subseteq \rangle$. The closure operations t, s, and r defined in Section 4.1 of the text are unary operations on $\mathscr{R}$, e.g. $t : \mathscr{R} \to \mathscr{R}$. Show that all three of these operations are monotonic.
Solution:

Transtive closure is monotonic:
First, we prove two lemmas:

Lemma 1 [Monotonicity of relation composition]: Let $R_1, R_2, R_3, R_4 \in \mathscr{P}(A \times A)$ such that $R_1 \subset R_3$ and $R_2 \subset R_4$. Then we have $(R_1 \circ R_2) \subset (R_3 \circ R_4)$.
Proof: If $a, c \in A$ and $(a,c) \in R_1 \circ R_2$, there is a $b \in A$ such that $(a,b) \in R_1$ and $(b,c) \in R_2$. Because $R_1 \subset R_3$ and $R_2 \subset R_4$, $(a,b) \in R_3$ and $(b,c) \in R_4$. Therefore, $(a,c) \in R_3 \circ R_4$. So we have $(R_1 \circ R_2) \subset (R_3 \circ R_4)$.

Lemma 2: If $R_1 \subset R_2$, then $R_1^k \subset R_2^k$ for all $k \geq 1$.
Proof: By mathematical induction.
Case k = 1: This holds immediately, since $R_1^k = R_1 \subset R_2 = R_2^k$.
Case k = n+1: Assume the induction hypothesis $R_1^n \subset R_2^n$. Since $R_1^k = R_1 \circ R_1^n$ and $R_2^k = R_2 \circ R_2^n$, the conclusion $R_1^k \subset R_2^k$ follows from the hypothesis $R_1 \subset R_2$ and the induction hypothesis by Lemma 1.

Now it follows from Lemma 2 and basic properties of union that if $R_1 \subset R_2$, then $t(R_1) = \bigcup_{i=1}^{\infty} R_1^i \subset \bigcup_{i=1}^{\infty} R_2^i = t(R_2)$. Therefore, transtive closure is monotonic.

Symmetric closure is monotonic:
Assume $R_1 \subset R_2$. Since $s(R_i) = R_i \cup R_i^{-1}$ for $i = 1, 2$, it suffices to show that $R_1^{-1} \subset R_2^{-1}$. Assume $(a,b) \in R_1^{-1}$. Then $(b,a) \in R_1$, and from the assumption $R_1 \subset R_2$ it follows that $(b,a) \in R_2$. Hence $(a,b) \in R_2^{-1}$, and it follows that $R_1^{-1} \subset R_2^{-1}$.

Reflexive closure is monotonic:
Since $r(R_i) = R_1 \cup I$ for $i = 1, 2$, $R_1 \subset R_2$ implies $r(R_1) \subset r(R_2)$ by the monotonicity of union.

6. [15] Consider the set of partitions of a set A, ordered by the refinement order: $P_1 \preceq P_2$ iff $\forall X \in P_1 . \exists Y \in P_2 . X \subseteq Y$. Show that every set of partitions of A has a lub and a glb. [**Hint**: consider the meaning of the lub and glb in terms of the equivalence relations associated with the partitions.]
Solution:
Let $\mathscr{M} = \{P_i\}$ be a set of partitions of A, and let $\mathscr{M}_R = \{R_i\}$ be the set of associated equivalence relations on A, i.e. R_i is the equivalence relation derived from P_i.

Lemma: If R_1 and R_2 are two equivalence relations of A, and P_1 and P_2 are the corresponding partions, then $P_1 \preceq P_2$ if and only if $R_1 \subset R_2$.
Proof: Assume that $P_1 \preceq P_2$, and that $(a,b) \in R_1$. Then $a, b \in X$ for some $X \in P_1$, namely $X = [a]_{R_1}$. Because $P_1 \preceq P_2$, there is a $Y \in P_2$ such that $X \subset Y$. So we have $a, b \in Y$ and therefore $(a,b) \in R_2$. Therefore, $R_1 \subset R_2$.
Assume conversely that $R_1 \subset R_2$. Let $X \in P_1$. Then $X = [a]_{R_1}$ for some a. Let $Y = [a]_{R_2}$. Then for any $b \in X$ we have $(a,b) \in R_1$ and hence $(a,b) \in R_2$, and hence $b \in [a]_{R_2}$, or $b \in Y$. Thus $X \subset Y$, and it follows that $P_1 \preceq P_2$.

For the problem:
The lub part:
Let $R = \bigcup_i R_i$. Let $E_u = tsr(R)$, the smallest equivalence relation containing R. (Actually, R will be reflexive and symmetric, so we really only need to take the transitive closure.) E_u is an equivalence relation and we claim that the partition P_u associated with E_u is the lub of $\mathscr{M}$. It is an upper bound: For any $R_i \in \mathscr{M}_R$, $R_i \subset E_u$. Therefore, $P_i \preceq P_u$.
It is the lub: If partition P' is an upper bound of $\mathscr{M}$ and E' is the associated partition, then by the Lemma, $E' \supset R_i$ for all i. But E_u is the least equivalence relation containing all the relations R_i, so $E_u \subset E'$, and hence $P_u \preceq P'$.

The glb part:
Let $E_l = \bigcap_i R_i$. We know that the set of equivalence relations is closed under intersection, so E_l is an equivalence relation. We claim that the partition P_l associated with E_l is the glb of $\mathscr{M}$.
It is a lowerbound: For any $R_i \in \mathscr{M}_R$, $R_i \supset E_l$. Therefore, the $P_l \preceq P_i$. It is the glb: If P' is a lower bound of $\mathscr{M}$, then by the Lemma, the associated equivalence relation E' is a lower bound for $\mathscr{M}_R$, i.e. $E' \subset R_i$ for all i. But then $E' \subset E_l$, so $P' \preceq P_l$. Hence P_l is the glb of $\mathscr{M}$.

7. [10] Given a poset $\langle A, \ \leq \rangle$, a subset $C \subseteq A$ is a *co-chain* if no two elements in C are comparable (i.e. related by $\leq$).
(a) For the set $\mathscr{P}(\{a,b,c,d\})$ ordered by subset, give the largest maximal co-chain (a maximal co-chain is a co-chain that is not a proper subset of a larger co-chain).
(b) Give an example of a poset with an infinite co-chain.
Solution:
(a)
The largest co-chain is the set of all two-element subsets, $\{\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}\}$, which has 6 elements.
(b) Let $\langle A, \leq \rangle$ be a poset where $A = \omega$ and $\leq$ be the identity relation I. Then $A = \{0,1,2,3,...\}$ itself is an infinite co-chain. This ordering is called the *discrete* ordering, and is the smallest possible ordering on A. Of course, any other infinite set with the discrete ordering would work.

8. [10] Show that if A is an infinite set, then $\mathscr{P}(A)$ is not well-founded.
Solution: If A is an infinite set, we may pick an infinite sequence of elements $a_1, a_2, ..., a_i, ...$ of A. Let $B_0 = A$ and $B_n = A - \{a_1, a_2, ..., a_n\}$. $B_i \in \mathscr{P}(A)$ and we have an infinite descending chain $B_0 \succeq B_1 \succeq B_2 \succeq ....$ Therefore, $\mathscr{P}(A)$ is not well-founded.

9. [10] A preorder is a relation $R \subseteq A \times A$ such that R is reflexive and transitive. It should be clear that given any binary relation Q on A, the reflexive, transitive closure $tr(Q)$ is a preorder. Show that given any preorder R on A, there is an equivalence relation $\sim_R$ on A such that $R/\!\sim_R$ is a partial order on $A/\!\sim_R$, where $R/\!\sim_R = \{([a],[b]) \mid R(a,b)\}$
Solution:
We define $E = \sim_R$ as follows: For $a, b \in A$, $E(a,b)$ if and only if $R(a,b) \wedge R(b,a)$.

E is reflexive: For $a \in A$, we have $R(a,a)$, therefore we have $E(a,a)$.

E is symmetric: For $a, b \in A$, if we have $E(a,b)$, we have $R(a,b)$ and $R(b,a)$. Therefore, we have $E(b,a)$.

E is transitive: For $a,b,c \in A$, if we have $E(a,b)$ and $E(b,c)$, we have $R(a,b)$, $R(b,a)$, $R(b,c)$, and $R(c,b)$. Because R is transitive, we have $R(a,c)$ and $R(c,a)$. Therefore, we have $E(a,c)$.

Thus E is an equivalence relation.

Next we have to prove that R/E is a well-defined relation on the quotient set A/E and that R/E is a partial order.

R/E is well-defined:
Let $a,b,c,d \in A$ such that $E(a,c)$ and $E(b,d)$. We need to show that $R(a,b)$ if and only if $R(c,d)$. If $R(a,b)$, because $R(c,a)$, $R(b,d)$ (by the assumption and the definition of E), and R is transitive, we have $R(c,d)$. If $R(c,d)$, because $R(a,c)$, $R(d,b)$ (by the assumption and the definition of E), and R is transitive, we have $R(a,b)$.

R/E is a partial order:
If $a \in A$, $([a],[a]) \in R/E$ because $(a,a) \in R$. Therefore, R/E is reflexive. If $a,b,c \in A$, $([a],[b]) \in R/E$, and $([b],[c]) \in R/E$, by definition, $(a,b) \in R$ and $(b,c) \in R$. Because R is transitive, $(a,c) \in R$. So we have $([a],[c]) \in R/E$. Therefore, R/E is transitive. If $a,b \in A$, $([a],[b]) \in R/E$, and $([b],[a]) \in R/E$, by definition, $(a,b) \in R$ and $(b,a) \in R$. By the definition of E, $[a] = [b]$. Therefore, R/E is antisymmetric. Thus, R/E is a partial order.