CS 153	**Foundations of Software**	**Homework Solution 2**
Fall 2006		**Due October 13, 2006**

1. Exercise 2.5.3(b,d), p. 125 [5 points]

(b) Let L be the set of all lists over $\{a,b\}$. Since all lists are finite (*i.e.*, have finite length), L can be expressed as the countable union $L = \cup_{n=1}^{\infty} L_n$, where L_n is the set of lists of length n. Each L_n is finite (in fact $|L_n| = 2^n$) and therefore countable, so by 2.10 L is countable.

(d) For each $i \in \mathbb{N}$, let $N_i = \{(i,j,k) | j \in \mathbb{N},\ k \in \mathbb{N}\}$. The function $f_i : N_i \to \mathbb{N} \times \mathbb{N}$ defined by $f_i(i,j,k) = (j,k)$ is clearly a bijection, and this establishes that N_i is countable for each $i \in \mathbb{N}$. The countability of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ follows from the fact that it is the countable union of the sets N_i for $i \in \mathbb{N}$:

$$\mathbb{N} \times \mathbb{N} \times \mathbb{N} = \cup_{i \in \mathbb{N}} N_i$$

Here is an alternate direct proof of the countability of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Cantor's pairing function $p : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection. Define the function $q : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by:

$$q(i,j,k) = p(p(i,j),k)$$

Claim: q is a surjection.
For any $m \in \mathbb{N}$, there exists a pair (n,k) such that $p(n,k) = m$, since p is surjective. For the same reason, there exist (i,j) such that $p(i,j) = n$. Then $q(i,j,k) = p(n,k) = m$, so q is surjective.

Claim: q is injective.
Suppose $q(a,b,c) = q(d,e,f)$. Then $p(p(a,b),c) = p(p(d,e),f)$, and since p is injective, this implies $p(a,b) = p(d,e)$ and $c = f$. Again we use the fact that p is injective to conclude $a = d$ and $b = e$. It follows that $(a,b,c) = (d,e,f)$, so q is injective.

Hence q is a bijection, and hence $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is countable.

2. **Exercise 2.5.6**, p.125 [5 points]

If A is uncountable and $B \subset A$ is countable, then $A - B$ is uncountable.

Proof by contradiction:

Assume $A - B$ is countable. $A = B \cup (A - B)$. But then, by 2.10 A is a union of a countable set of countable sets and is therefore countable.

3. **Exercise 2.5.10**, p.125 [10 points]

We are to prove $|A| < |\mathscr{P}(A)|$.

Proof (by contradiction):

Each element $x \in A$ can be mapped to the singleton $\{x\} \in \mathscr{P}(A)$, and this mapping is obviously injective, thus establishing that $A \preceq \mathscr{P}(A)$, or, equivalently, $|A| \leq |\mathscr{P}(A)|$.

Assume $|A| = |\mathscr{P}(A)|$, i.e. there is a bijection f from A to $\mathscr{P}(A)$. Then, each $x \in A$ is associated with the subset $f(x) \subseteq A$. Define

$$S = \{x \in A \mid x \notin f(x)\}$$

Since $S \subseteq A$ and f is a bijection, there is a $y = f^{-1}(S) \in A$ such that $f(y) = S$. If $y \in S$, then by the defining property of S, $y \notin f(y)$, i.e., $y \notin S$, a contradiction. On the other hand, if $y \notin S$, then $y \notin f(y)$, so $y \in S$, another contradiction. Hence we arive at a contradiction whether we assume $y \in S$ or the opposite. Therefore the assumption that the bijection f exists and $|A| = \mathscr{P}(A)$ must be false.

Since $|A| \leq |\mathscr{P}(A)|$ is established by the injection $x \mapsto \{x\}$, we must have $|A| \leq |\mathscr{P}(A)|$.

4. $Nat \sim Nat \times Nat$ [10 points]

By Schröder-Bernstein, it suffices to find two injections, $f : \mathbb{N} \hookrightarrow Nat \times Nat$ and $g : \mathbb{N} \times Nat \hookrightarrow Nat$. Take (for example), f to be defined by $f(n) = (n,0)$, and g to be defined by $g(n,m) = 2^n 3^m$. These are clearly injections, so we have $Nat \preceq Nat \times Nat$ and $Nat \times Nat \preceq Nat$, and hence $Nat \sim Nat \times Nat$ by Schröder-Bernstein.

5. **Exercise 4.1.2**, p.210 [5 points] (b): reflexive, symmetric, transitive (i.e., an equivalence)
(d): reflexive, antisymmetric, transitive (i.e., a partial order)

6. **Exercise 4.1.7**, p.211 [5 points]

Let $A = \{a,b,c\}$ and let $R \subseteq A \times A$ be the relation given by

$$R = \{(a,a),(a,b),(b,c),(c,a)\}$$

Then R is antisymmetric, by inspection. But

$$R^2 = \{(a,a),(a,b),(b,a),(a,c),(c,a),(c,b)\}$$

is not antisymmetric (we have $(a,b) \in R$ and $(b,a) \in R$ but $a \neq b$).

7. **Exercise 4.1.13**, p.211 [5 points]
(b) $\{(a,b),(a,c),(b,c)\}$ (i.e. already closed)
(d) all pairs: $A \times A$ where $A = \{a,b,c,d\}$

8. **Exercise 4.1.25**, p.213 [5 points]
Show that $\forall R \subseteq A \times A.\ r(s(A)) = s(r(A))$.

Proof: This can be proved routinely by reasoning about elements, but here we will do an "algebraic" proof. First we note some general algebraic laws satisfied by union, the relational inverse operation, and the identity relation on A (which we will denote by I).

$A \cup (B \cup C) = (A \cup B) \cup C$	Associative union	(1)
$A = A \cup A$	Idempotent union	(2)
$I^{-1} = I$	I self-inverse	(3)
$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$	Inverse distributes over union	(4)

These equations all have very simple, straightforward proofs. Now we can exploit these equational laws to prove the proposition.

$$\begin{aligned}
r(s(R)) &= I \cup (R \cup R^{-1}) && \text{(defns of s and r)} \\
&= (I \cup I) \cup (R \cup R^{-1}) && \text{(idempotence of union)} \\
&= (I \cup R) \cup (I \cup R^{-1}) && \text{(associativity of union)} \\
&= r(R) \cup (I \cup R^{-1}) && \text{(defn of r)} \\
&= r(R) \cup (I^{-1} \cup R^{-1}) && \text{(I self inverse)} \\
&= r(R) \cup (I \cup R)^{-1} && \text{(inverse distributes over union)} \\
&= r(R) \cup (r(R))^{-1} && \text{(defn of r)} \\
&= s(r(R)) && \text{(defn of s)}
\end{aligned}$$

9. **Exercise 4.2.4**, p.230 [5 points]

Let $f(x) = floor(x)$, so $f : \mathbb{R} \to \mathbb{N}$. The equivalence class of $x \in \mathbb{R}$ relative to the kernel relation for f is given by the semiclosed interval $[f(x), f(x)+1)$, i.e.

$$[x]_f = [f(x), f(x)+1)$$

10. **Exercise 4.2.8**, p.231 [10 points]

Proof:

We are given that binary relation R on S is symmetric and transitive, and for each $x \in S$ there is $y \in S$ s.t. $(x,y) \in R$. By symmetry, $(y,x) \in R$, too. Then by transitivity, $(x,x) \in R$. Thus, R is reflexive.

11. **Exercise 4.2.13**, p.231 [10 points] Given $f : A \to B$, let P be the partition of A consisting of the equivalence classes of A relative to the kernel equivalence relation induced by f. Then we can define the two functions

$$\begin{aligned}
s &: A \to P, \quad s(a) = [a] \\
i &: P \to B, \quad i([a]) = f(a)
\end{aligned}$$

and we clearly have $f = i \circ s$. We need to show that s is a surjection and i is a injection.

By the definition of P, any element $X \in P$ is of the form $X = [a]$ for some $a \in A$, and hence $X = s(a)$, and so s is surjective.

Suppose $i([a]) = i([b])$. Then by the definition of i it follows that $f(a) = f(b)$, which implies that $[a] = [b]$. Therefore i is injective.

12. [15 points]

$R \in A \times A$ a binary relation. The set of equivalence relations on A containing R is:

$$\mathscr{E} = \{E \subseteq A \times A \mid E \text{ is an equivalence relation on } A \ \wedge \ R \in E\}$$

Claim: $\mathscr{E}$ not empty and $\cap \mathscr{E} = tsr(R)$.

Proof:

The "complete" relation $A \times A$ is trivially an equivalence relation on A and it certainly contains R, so $A \times A$ is a member of $\mathscr{E}$ and therefore $\mathscr{E}$ is not empty. Let $E_0 = \cap\mathscr{E}$. By Prop 4.16, $tsr(R)$ is an equivalence relation and is contained in every equivalence relation E that contains R, so $tsr(R) \subseteq E_0$.

On the other hand, since $tsr(R)$ is an equivalence relation containing R, by the definition of $\mathscr{E}$ we conclude that $tsr(R) \in \mathscr{E}$. Since the intersection of a collection of sets is a subset of every set in the collection, we have $E_0 \subseteq tsr(R)$. It follows from this and the conclusion of the previous paragraph that $tsr(R) = E_0$.

13. **Exercise 4.3.7**, p.252 [5 points]

Let $leaves : tree \to \mathbb{N}$ be the function that returns the number of leaves of a tree. We assume the ordering on trees is given by

$$s \prec t \;\Leftrightarrow\; leaves(s) < leaves(t)$$

It follows that if there were an infinite descending chain $t_0 \succ t_1 \succ t_2 \succ \ldots$, then the images under *leaves* would form an infinite descending chain of natural numbers $leaves(t_0) > leaves(t_1) > leaves(t_2) > \ldots$, which can't happen since $\mathbb{N}$ is well-founded. Therefore one cannot have an infinite descending chain of trees, and $\prec$ is well founded.

14. **Exercise 4.3.12**, p.252 [5 points]

Claim: R irreflexive and transitive implies R is antisymmetric.

Proof: Suppose for some a and b we had $(a,b) \in R$ and $(b,a) \in R$. Then since R is transitive we must have $(a,a) \in R$, which contradicts the assumption that R is irreflexive. Hence there are no such elements a and b, and antisymmetry holds vacuously, since its premise is always false.

15. [10] Consider the relation of refinement on partitions of a set A as a partial order. Show that if A is infinite, the partial order of partitions is not well-founded.

Since A is infinite, $\mathbb{N} \preceq A$, meaning that there is an injection $f : \mathbb{N} \hookrightarrow A$. Let $a_i = f(i)$ for each $i \in \mathbb{N}$ and note that the elements a_i are all distinct. Now we define a family of partitions P_i of A as follows:

$$\begin{aligned}
A_0 &= \emptyset \\
A_{n+1} &= A_n \cup \{a_n\} \\
P_0 &= \{A\} \\
P_n &= \{A - A_n, \{a_0\}, \ldots, \{a_{n-1}\}\} \quad (n > 0)
\end{aligned}$$

Then if $\prec$ is the refinement ordering on partitions of A, we have an infinite descending chain $P_0 \succ P_1 \succ P_2 \succ \ldots$, implying that $\prec$ is not well-founded.