CS 153	Foundations of Software	Homework Solution 1
Fall 2006		Due October 4, 2006

1. Exercise 1.2.13(b) (Section 1.2, Exercise 13(b), p. 32) [5 points]
Solution:
$(B \cup C) - (A \cap C)$

2. 1.2.25 (p. 34) [10]
Solution:
For any x, we have

$$\begin{aligned} x \in A \cup (B \cap C) &\Leftrightarrow x \in A \vee (x \in B \cap C) \quad \text{[defn } \cup\text{]} \\ &\Leftrightarrow x \in A \vee (x \in B \wedge x \in C) \quad \text{[defn } \cap\text{]} \\ &\Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \quad \text{[logic]} \\ &\Leftrightarrow (x \in A \cup B) \wedge (x \in A \cup C) \quad \text{[defn } \cup\text{]} \\ &\Leftrightarrow x \in (A \cup B) \cap (A \cup C) \quad \text{[defn } \cap\text{]} \end{aligned}$$

Therefore $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ have the same members, so they are equal sets.

3. 1.2.18 (p. 33) [10]
Solution:
We classify people in the summer camp according to the following table:

gender/adulthood	boy		girl		man		woman	
city or noncity	city	noncity	city	noncity	city	noncity	city	noncity
total number	a	b	c	d	e	f	g	h

The conditions given in the problem can be expressed by the following equations:

$$\begin{aligned} a+b &= 27 \\ a+c &= 15 \\ e+f &= 27 \\ b &= 21 \\ a+c+e+g &= 42 \\ a+e &= 18 \\ d+h &= 21 \end{aligned}$$

Solving these equations, we get the values $a = 6$, $b = 21$, $c = 9$, $e = 12$, $f = 15$, and $g = 15$, which, together with the fact that $d + h = 21$, gives us: $a + b + c + d + e + f + g + h = 99$ (the total number of people).

4. 1.2.29(d) (p. 35) [10]
Solution:
For any x, we have

$$\begin{aligned} x \in (A \cup B)' &\Leftrightarrow \neg(x \in (A \cup B)) \qquad \text{[defn ']} \\ &\Leftrightarrow \neg((x \in A) \vee (x \in B)) \qquad \text{[defn } \cup\text{]} \\ &\Leftrightarrow \neg(x \in A) \wedge \neg(x \in B) \qquad \text{[logic]} \\ &\Leftrightarrow (x \in A') \wedge (x \in B') \qquad \text{[defn ']} \\ &\Leftrightarrow x \in (A' \cap B') \qquad \text{[defn } \cap\text{]} \end{aligned}$$

Therefore $(A \cup B)'$ and $A' \cap B'$ have the same members, so they are equal sets.

5. 1.2.30 (p. 35) [10]
Solution:
None of the normal set constructions allow us to construct a set which is a member of itself, and we have no mechanism for solving *recursive* set equations of this sort. Therefore it does not appear to be possible to build a set A satisfying the equation $A = \{a, A, b\}$. [In fact, one of the axioms in the most common axiomatic formulation of set theory due to Zermelo and Fraenkel (and commonly called ZF Set Theory) is the Axiom of Foundation, which states that the membership relation $\in$ is a well-founded relation (i.e. there are no infinite descending chains where each element is a member of the predecessor, like $\ldots A_n \in A_{n-1} \in \ldots A_2 \in A_1 \in A_0$). If A is a set that is a member of itself, $\ldots A \in A \in A$ forms such an infinite descending chain, and violates the Axiom of Foundation.

6. 1.3.19(b) (p. 54) [10]
Solution:
$(x, y) = (u, v) \Rightarrow x = u \wedge y = v$:

> If $x = y$, $(x, y) = \{\{x\}\}$, a singleton. Therefore, $(u, v) = \{\{u\}, \{u, v\}\}$ must be a singleton, i.e., $u = v$ and $(u, v) = \{\{u\}\}$. Then $\{\{x\}\} = \{\{u\}\} \Rightarrow \{x\} = \{u\} \Rightarrow x = u$. We also have $y = v$ since $y = x$ and $v = u$. If $x \neq y$, then (x, y) is not a singleton, so (u, v) is not a singleon and hence $u \neq v$. We have $\{\{x\}, \{x, y\}\} = \{\{u\}, \{u, v\}\}$, which implies that the singleton sets $\{x\}$ and $\{u\}$ are equal and the two element sets $\{x, y\}$ and $\{u, v\}$ are equal. Now $\{x\} = \{u\}$ implies $x = u$, and $\{x, y\} = \{u, v\}$ implies $y = v$ since $y \in \{u, v\}$ and we know $y \neq x$ which implies $y \neq u$.

$x = u \wedge y = v \Rightarrow (x, y) = (u, v)$:

$$\begin{aligned} (x, y) &= \{\{x\}, \{x, y\}\} \\ &= \{\{u\}, \{u, v\}\} \quad \text{(by substitution of equals for equals)} \\ &= (u, v) \end{aligned}$$

7. 2.1.12(b) (p. 89) [5]
Solution:
$\chi_{A \cap B}(x) = \chi_A(x)\chi_B(x)$

8. 2.1.19(b,d) (p. 90) [10]
Solution:
(b) For any x, we have

$$\begin{aligned} x \in f^{-1}(G \cap H) &\Leftrightarrow f(x) \in (G \cap H) \qquad \text{[defn of inverse image } f^{-1}\text{]} \\ &\Leftrightarrow (f(x) \in G) \wedge (f(x) \in H) \qquad \text{[defn } \cap\text{]} \\ &\Leftrightarrow (x \in f^{-1}(G)) \wedge (x \in f^{-1}(H)) \qquad \text{[defn } f^{-1}\text{]} \\ &\Leftrightarrow x \in (f^{-1}(G) \cap f^{-1}(H)) \qquad \text{[defn } \cap\text{]} \end{aligned}$$

Therefore $f^{-1}(G \cap H)$ and $f^{-1}(G) \cap f^{-1}(H)$ have the same members, so they are equal sets.

(d) If $x \in f(f^{-1}(G))$, there is a $y \in f^{-1}(G)$ such that $x = f(y)$. But $y \in f^{-1}(G)$ means that $f(y) \in G$. Therefore, $x \in G$.

9. 2.3.15(b) (p. 114) [5]
Solution:
Let $f : A \to B$ and $g : B \to C$ be surjective functions. For any $c \in C$, because g is surjective, there is a $b \in B$ such that $g(b) = c$. Because f is surjective, there is an $a \in A$ such that $f(a) = b$. $(g \circ f)(a) = g(f(a)) = g(b) = c$. Therefore, $g \circ f$ is surjective.

10. [5] Give an example of a (partial) function (a) from $\emptyset$ to $\mathbf{nat}$; (b) from $\mathbf{nat}$ to $\emptyset$.
Solution:
(a) A function from $\emptyset$ to $\mathbf{nat}$ is a subset of the cartesian product $\emptyset \times \mathbf{nat}$, which is $\emptyset$ (there are no ordered pairs whose first element is a member of $\emptyset$). Therefore the function is also $\emptyset$, since that is the only subset of $\emptyset$. (b) By the same reasoning as in part (a), the only function is the empty function given by $\emptyset$.

11. [20] Show that the intersection and union of two binary relations on $A \times B$ are relations. Show that the complement of a relation is a relation. Is the intersection of two functions a function? Is the union of two functions a function? Is the complement of a function a function? (In each case, prove it or give a counterexample.)
Solution:
By definition, a binary relation on $A \times B$ is simply a subset of $A \times B$. Therefore, if R_1 and R_2 are relations on A and B, then $R_1 \subset A \times B$ and $R_2 \subset A \times B$, so $R_1 \cap R_2 \subset A \times B$ and $R_1 \cup R_2 \subset A \times B$ and hence they are relations on $A and$B. Furthermore, $R_1' \subset A \times B$, assuming we are taking the complement relative to $A \times B$, so it too is a relation on A and B.

Let f_1 and f_2 be functions in $A \to B$, i.e., single-valued relations from A to B. Let $g = f_1 \cap f_2$. From the above discussion, g is a relation from A to B. If g is not single-valued, then there are pairs $(x, y_1) \in g$ and $(x, y_2) \in g$ with $y_1 \neq y_2$. Since g is a subset of both f_1 and f_2, these two pairs will also be in f_1 and f_2, contradicting the assumption that they are functions. Therefore g must be single-valued. So the intersection of two functions is a function.

On the other hand, the union of two functions may not be a function. Consider $A = B = \{a, b\}$ and functions $f_1 = \{(a, a), (b, a)\}$ and $f_2 = \{(a, b), (b, b)\}$. The union of these two functions is the relation $\{(a, a), (b, a), (a, b), (b, b)\}$, which is not single-valued.

The complement of a function may also fail to be a function. Consider the function $f : a \to \{1, 2, 3\}$ given by $f = \{(a, 1\}$. Its complement is $\{(a, 2), (a, 3)\}$, which is not a function.