CS 153	Foundations of Software	Handout 2
Fall 2006		11/17/06

Sequent Calculus for Natural Deduction

Reference: *Logic and Computation* by L. C. Paulson, Cambridge University Press.

Judgements, or logical assertions to be proved, are of the form $\Gamma \vdash A$, where A is a formula, and Γ is a set or sequence of formulas: $\Gamma = A_1, A_2, \ldots A_n$. Intuitively, the judgement $\Gamma \vdash A$ asserts that A is true, or more precisely, provable, under the hypotheses Γ. We also use other capital Greek letters like Δ and $Theta$ for formula sets, and we use Γ, Δ to denote the union of Γ and Δ.

Assumption: $A \vdash A$

Conjunction:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \ (\wedge \text{ intro})$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \ (\wedge \text{ elim 1}) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \ (\wedge \text{ elim 2})$$

$$\frac{\Gamma \vdash A \wedge B \qquad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C} \ (\wedge \text{ elim 3})$$

Disjunction:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \ (\vee \text{ intro 1}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \ (\vee \text{ intro 2})$$

$$\frac{\Gamma \vdash A \vee B \qquad \Delta, A \vdash C \qquad \Theta, B \vdash C}{\Gamma, \Delta, \Theta \vdash C} \ (\vee \text{ elim})$$

Implication:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \ (\rightarrow \text{ intro}) \qquad \frac{\Gamma \vdash A \rightarrow B \qquad \Delta \vdash A}{\Gamma, \Delta \vdash B} \ (\rightarrow \text{ elim, MP})$$

Negation:

$$\frac{\Gamma, A \vdash \mathbf{F}}{\Gamma \vdash \neg A} \ (\neg \text{ intro}) \qquad \frac{\Gamma \vdash \neg A \qquad \Delta \vdash A}{\Gamma, \Delta \vdash \mathbf{F}} \ (\neg \text{ elim})$$

Contradiction:

$$\frac{\Gamma, \neg A \vdash \mathbf{F}}{\Gamma \vdash A} \ (\text{F elim, classical}) \qquad \frac{\Gamma \vdash \mathbf{F}}{\Gamma \vdash A} \ (\text{F elim, int.})$$

The additional rules for the quantifier forms in the predicate calculus are as follows:

Universal Quantifier:

$$\dfrac{\Gamma \vdash A}{\Gamma \vdash \forall x.A} \quad x \text{ not free in } \Gamma \quad (\forall \text{ intro})$$

$$\dfrac{\Gamma \vdash \forall x.A}{\Gamma \vdash A[t/x]} \quad (\forall \text{ elim})$$

Existential Quantifier:

$$\dfrac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x.A} \quad (\exists \text{ intro})$$

$$\dfrac{\Gamma \vdash \exists x.A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B} \quad x \text{ not free in } \Delta, B \quad (\exists \text{ elim})$$

Example 1 (Exercise 6.3.5(b) from Homework 6). The proof of $A \to (\neg B \to (A \wedge \neg B))$:

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad \neg B \vdash \neg B}{A, \neg B \vdash A \wedge \neg B}\ (\wedge \text{ intro})}{A \vdash \neg B \to (A \wedge \neg B))}\ (\to \text{ intro})}{\vdash A \to (\neg B \to (A \wedge \neg B))}\ (\to \text{ intro})$$

Example 2 (Exercise 6.3.5(d) from Homework 6). The proof of $(B \to C) \to (A \wedge B \to A \wedge C)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{(A \wedge B) \vdash A \wedge B}{(A \wedge B) \vdash A}\ (\wedge \text{ elim}) \qquad \dfrac{(B \to C) \vdash B \to C \qquad \dfrac{(A \wedge B) \vdash A \wedge B}{(A \wedge B) \vdash B}\ (\wedge \text{ elim})}{(B \to C), (A \wedge B) \vdash C}\ (\to \text{ elim})}{(B \to C), (A \wedge B) \vdash A \wedge C}\ (\wedge \text{ intro})}{(B \to C) \vdash A \wedge B \to A \wedge C}\ (\to \text{ intro})}{\vdash (B \to C) \to (A \wedge B \to A \wedge C)}\ (\to \text{ intro})$$

Example 3 To prove $\neg \forall x.A \vdash \exists x.\neg A$ we need first to prove some derived rules.

(R1) $\dfrac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$

$$\dfrac{\dfrac{\Gamma \vdash \neg\neg A \qquad \dfrac{}{\neg A \vdash \neg A}}{\Gamma, \neg A \vdash \mathbf{F}}\ (\neg \text{ elim})}{\Gamma \vdash A}\ (\mathbf{F} \text{ elim})$$

(R2) $\dfrac{\Gamma \vdash \forall x.\neg\neg A}{\Gamma \vdash \forall x.A}$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \forall x.\neg\neg A}{\Gamma \vdash \neg\neg A}\ (\forall \text{ elim})}{\Gamma \vdash A}\ (\text{R1})}{\Gamma \vdash \forall x.A}\ (\forall \text{ intro})$$

(R3) $\neg(A \vee B) \vdash \neg B$

$$\cfrac{\cfrac{\neg(A \vee B) \vdash \neg(A \vee B) \qquad \cfrac{B \vdash B}{B \vdash A \vee B}\;(\vee\text{ intro 2})}{\neg(A \vee B), B \vdash \mathbf{F}}\;(\neg\text{ elim})}{\neg(A \vee B) \vdash \neg B}\;(\neg\text{ intro})$$

(R4) $\neg(\exists x.A) \vdash \forall x.\neg A$

$$\cfrac{\cfrac{\cfrac{\neg\exists x.A \vdash \neg\exists x.A \qquad \cfrac{A \vdash A}{A \vdash \exists x.A}\;(\exists\text{ intro})}{\neg\exists x.A, A \vdash \mathbf{F}}\;(\neg\text{ elim})}{\neg\exists x.A \vdash \neg A}\;(\neg\text{ intro})}{\neg\exists x.A \vdash \forall x.\neg A}\;(\forall\text{ intro})$$

(swap) $\dfrac{\Gamma, \neg B \vdash A}{\Gamma, \neg A \vdash B}$

$$\cfrac{\cfrac{\Gamma, \neg B \vdash A \qquad \neg A \vdash \neg A}{\Gamma, \neg B, \neg A \vdash \mathbf{F}}\;(\neg\text{ elim})}{\Gamma, \neg A \vdash B}\;(\mathbf{F}\text{ elim})$$

And now we can finally prove $\neg\forall x.A \vdash \exists x.\neg A$, starting with (R4) instantiated with $\neg A$ for A.

$$\cfrac{\cfrac{\neg\exists x.\neg A \vdash \forall x.\neg\neg A}{\neg\exists x.\neg A \vdash \forall x.A}\;(\text{R2})}{\neg\forall x.A \vdash \exists x.\neg A}\;(\text{swap})$$