CS38600-1 Complexity Theory A **Spring 2003**

Lecture 4: Representations of $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$

Lecturer: Ketan Mulmuley *Scribe: Cameron Freer*

4.1 Introduction

We recall Theorem 1.8:

Theorem 4.1 (Complete Reducibility – Weyl)
Every finite dimensional representation of $GL_n(\mathbb{C})$ admits a unique decomposition into irreducibles:

$$W = \bigoplus_i V_i^{m_i} \tag{4.1}$$

Thus we wish to find the irreducible representations of $GL_n(\mathbb{C})$.

4.2 Irreducible Representations of $GL_n(\mathbb{C})$: First Construction (Deyruts)

Fix λ, a Young diagram of height $\leq n$. We associate with λ an irreducible representation V_λ (known as the Weyl module or Schur module). Let x be a variable square matrix of size n, on which $GL_n(\mathbb{C})$ acts by multiplication on the left. This induces a $GL_n(\mathbb{C})$-action on $\mathbb{C}[x]$ by $(A \cdot f)(x) = f(A^t x)$[1]. Hence $C[x]$ is a representation of $GL_n(\mathbb{C})$ (infinite dimensional).

For any column $c = \begin{pmatrix} c_1 \\ \vdots \\ c_l \end{pmatrix}$, where $c_i \in 1, ..., n$ are distinct, and $c_1 < c_2 < ... < c_l$, we define

$$e_c = \det \left(x^{(1 \ldots l)}_{(c_1 \ldots c_l)} \right),$$

i.e., the determinant of the minor of x with columns 1 to l and rows c_1 to c_l.

For any semistandard tableau T of shape λ, we define $e_T = \prod_c e_c$, where c ranges over the columns of T. Let $V_\lambda = \langle e_T \mid T \text{ semistandard of shape } \lambda \rangle$, i.e., the subspace of $\mathbb{C}[x]$ spanned by e_T's. To show that V_λ is invariant under $GL_n(\mathbb{C})$, it is enough to show that $A \cdot e_c = \sum_{c'} \alpha(c, c') \cdot e_{c'}$. To see that, we observe that $(A \cdot e_c)(x) = e_c(A^t x) = \det \left((A^t x)^{(1 \ldots l)}_{(c_1 \ldots c_l)} \right) = \sum_{c'} \alpha(c, c') \cdot e_{c'}$, from the properties of the determinant. Note that the c''s involved here of the same shape as c (i.e. same length). Since $A \cdot (fg) = (A \cdot f)(A \cdot g)$, it follows that $A \cdot e_T$ is also a linear combination of $e_{T'}$, where the T''s have the same shape as T. Hence we have that V_λ is also a representation of $GL_n(\mathbb{C})$.

Theorem 4.2 *1. V_λ is an irreducible representation of $GL_n(\mathbb{C})$ and also of $SL_n(\mathbb{C})$.*

[1] We need A^t for it to be an action

2. *Every irreducible finite dimensional representation of $SL_n(\mathbb{C})$ is isomorphic to V_λ for some λ of height $< n$. Furthermore, $V_\lambda \ncong V_{\lambda'}$ if $\lambda \neq \lambda'$.*

3. *Every irreducible finite dimensional representation of $GL_n(\mathbb{C})$ is isomorphic to $V_\lambda \otimes \mathrm{Det}^\alpha$, where* Det : $A \mapsto \det(A)$ *and $\alpha \in \mathbb{C}$. Similarly, $V_\lambda \otimes \mathrm{Det}^\alpha \ncong V_{\lambda'} \otimes \mathrm{Det}^{\alpha'}$ if $\lambda \neq \lambda'$ or $\alpha \neq \alpha'$.*

4. *The set $\{e_T \mid T$ semistandard of shape $\lambda\}$ is a basis of V_λ; i.e., the e_T's are linearly independent.*

Proof: (See Fulton & Harris, pp. 221-237.) Note that e_T is defined for any T of shape λ (not necessarily semistandard). To prove 4. we write an arbitrary R_T as a linear combination of semistandard e_T's. ■

The above result shows that the Det representation is the only non-trivial representation of $GL_n(\mathbb{C})$ which when restricted to $SL_n(\mathbb{C})$ becomes trivial.

4.3 Characters

Let W be a representation of $GL_n(\mathbb{C})$. Then define $\chi_W(g)$ to be the trace of g on W, where $g \in GL_n(\mathbb{C})$ (This is a class function that is invariant on conjugacy classes). From elementary linear algebra, we know that every matrix in $GL_n(\mathbb{C})$ is conjugate to a matrix in Jordan canonical form. Hence χ_W is determined by its values on matrices in Jordan canonical form. It is easy to see (by looking at the Jordan canonical form or otherwise) that the diagonalizable elements form a dense subset of $GL_n(\mathbb{C})$, so for any rational representation (i.e., the entries of the representation are rational functions of the entries of the matrix), χ_W is determined by its values on diagonal matrices $\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$, which we denote by $\chi_W(x_1, ..., x_n)$; this is called the character of W (and is a rational polynomial function, with a suitable power of the determinant as its denominator in the case of $GL_n(\mathbb{C})$).

Theorem 4.3 *$\chi_{V_\lambda}(x_1, ..., x_n) = S_\lambda(x_1, ..., x_n)$, and if $\lambda \neq \lambda'$, then $S_\lambda \neq S_{\lambda'}$.*

Proof:

Each e_T is an eigenvector of $\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$:

$$\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \cdot e_T = \prod_i \lambda_i^{\mu_i(T)} e_T,$$

where $\mu_i(T)$ is the number of i's in T. We call e_T a "weight vector" in the representation with weight $\mu = (\mu_1, ...\mu_n)$. Since $\{e_T\}$ is a basis, we have

$$\chi_W(x_1, ..., x_n) = \sum_T (\prod_i x_i^{\mu_i(T)}) = \sum_T \mathrm{Content}(T),$$

where we sum over semistandard T of shape λ, and $\mathrm{Content}(T) = S_\lambda(x_1, ..., x_n)$. ■

4.4 Irreducible Representations of $GL_n(\mathbb{C})$: Second Construction

Let $V = \mathbb{C}^n$, the standard representation of $GL_n(\mathbb{C})$. $GL_n(\mathbb{C})$ acts on $V^{\otimes d}$ from the left: $g(v_1 \otimes \cdots \otimes v_n) = g(v_1) \otimes \cdots \otimes g(v_n)$. S_d acts on $V^{\otimes d}$ from the right: $(v_1 \otimes \cdots \otimes v_n)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$. The two actions commute, so $V^{\otimes d}$ is a representation of $GL_n(\mathbb{C}) \times S_d$.

Each irreducible representation of $GL_n(\mathbb{C}) \times S_d$ is isomorphic to one of the form $U \otimes U'$, where U and U' are irreducible representations of $GL_n(\mathbb{C})$ and S_d, respectively. This is in turn isomorphic to $V_\lambda \otimes S^\mu$, where V_λ and S^μ are the corresponding Weyl and Specht modules, respectively. By Weyl's Theorem (4.1), we have

$$V^{\otimes d} = \sum_{\substack{\lambda, \mu \\ \mathrm{ht}(\lambda) \le n \\ \mathrm{size}(\mu) = d}} m_{\lambda,\mu} V_\lambda \otimes S^\mu \tag{4.2}$$

Next we give an explicit decomposition of $V^{\otimes d}$. Fix λ, with $\mathrm{size}(\lambda) = d$. Fix T to be any standard tableau of shape λ. Define the Young symmetrizer c_λ^T to be $a_\lambda b_\lambda$, where $a_\lambda = \sum_{g \in \mathrm{Row}(T)} e_g$ and $b_\lambda = \sum_{g \in \mathrm{Col}(T)} \mathrm{sgn}(g) e_g$. Let $S_\lambda^T(V) = V^{\otimes d} c_\lambda^T$, the image of $V^{\otimes d}$ under c_λ^T.

Theorem 4.4 *1. $S_\lambda^T(V) \cong S_\lambda^{T'}(V)$ if T and T' are standard of the same shape λ. If $\mathrm{ht}(\lambda) > n$, then $S_\lambda(V) = \{0\}$.*

2. $S_\lambda(V)$ is isomorphic to V_λ in Theorem 4.2; i.e., it is an irreducible representation of $GL_n(\mathbb{C})$ (if $\mathrm{ht}(\lambda) \le n$).

Example 4.5 *1. $T = (1...d)$. We have $c_\lambda : v_1 \otimes \cdots \otimes v_n \mapsto \sum_{\sigma \in S_d} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$, so $S_\lambda = \mathrm{Sym}^d(V)$.*

2. $T = \begin{pmatrix} 1 \\ \vdots \\ l \end{pmatrix}$. We have $c_\lambda : v_1 \otimes \cdots \otimes v_n \mapsto \sum_{\sigma \in S_l} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$, so $S_\lambda = \Lambda^l(V)$.

3. $\lambda = (2,1)$, $T = \begin{matrix} 1 & 2 \\ 3 & \end{matrix}$.
$c_\lambda = c_{(2,1)} = (e_1 + e_{(1\ 2)})(e_1 - e_{(1\ 3)}) = 1 + e_{(1\ 2)} - e_{(1\ 3)} - e_{(1\ 3\ 2)}$.
Thus $S_\lambda(V) =$ the image of c_λ on $V^{\otimes d} =$ the span of $v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 - v_3 \otimes v_1 \otimes v_2$.

Theorem 4.6 *1. As a representation of $GL_n(\mathbb{C})$, $V^{\otimes d} = \sum_{\lambda,T} S_\lambda^T(V)$, where λ ranges over Young diagrams of size d and T ranges over standard tableau of shape λ.*

2. $V^{\otimes d} = \sum_\lambda S_\lambda(V) \otimes S^\lambda(V)$, where again λ ranges over Young diagrams of size d, and here the first S term is the Weyl module and the second is the Specht module. In other words, in the formula (4.2), we have $m_{\lambda,\mu} = 1$ if $\mu = \lambda$, and $m_{\lambda,\mu} = 0$ if $\mu \ne \lambda$.

Corollary 4.7 *(to Theorem 4.3)*
The finite dimensional representations of $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$ are determined by their characters.

Proof: We have that $S_\lambda \neq S_{\lambda'}$ if $\lambda \neq \lambda'$ and that the S_λ are linearly independent. If V is any polynomial representation of $GL_n(\mathbb{C})$ (or $SL_n(\mathbb{C})$) with character $\chi_V(x_1, ..., x_n)$, then let (by Weyl's theorem) $v = \sum m_\lambda v_\lambda$. Then $\chi_V = \sum m_\lambda S_\lambda$. Therefore to calculate m_λ we just express the character χ_V in the Schur basis $\{S_\lambda\}$; the coefficients correspond to the multiplicities.

■

4.5 Tensor Products and a Decision Problem

Suppose we have $W = S_\lambda(V) \otimes S_\mu(V) = V_\lambda \otimes V_\mu$. How does $V_\lambda \otimes V_\mu$ decompose? Write $V_\lambda \otimes V_\mu = \sum_\nu N_{\lambda\mu\nu} V_\nu$, so that now the problem is reduced to computing the terms $N_{\lambda\mu\nu}$. We have $\chi_W = S_\lambda S_\mu$, so $S_\lambda S_\mu = \sum_\nu N_{\lambda\mu\nu} S_\nu$, which is a symmetric function, so we may express it in terms of the Schur basis. The Littlewood-Richardson rule gives a combinatorial property for computing the terms $N_{\lambda\mu\nu}$. Using identities in symmetric function theory, they showed that $N_{\lambda\mu\nu}$ is the number of ways in which the Young diagram λ can be expanded.

Problems

1. What is $N_{\lambda\mu\nu}$ explicitly?
2. Given β, is V_β a subset of $V_\lambda \otimes V_\mu$, i.e., is $N_{\lambda\mu\nu} \neq 0$?

Goal: A polynomial time algorithm for the decision problem "Does $N_{\lambda\mu\nu} = 0$?".