CS38600-1 Complexity Theory A **Spring 2003**

Lecture 6 : Kronecker Product of Schur Functions – Part I

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Abstract

The irreducible representations of S_n, i.e. the Specht modules are indexed by partitions λ of n. For any two partitions λ, μ of n, $S_\lambda \otimes S_\mu = g_{\lambda\mu\nu} S_\nu$, for suitable integers $g_{\lambda\mu\nu}$. The actual values of these coefficients still eludes us. We look at a formula (admittedly messy), which gives the exact values of $g_{\lambda\mu\nu}$ for simple shapes λ, μ.

6.1 Recall

Before we get into the main topic, let us recall what we know about the representations of S_n.

Let G be any finite group, and $\mathbb{C}[G]$ the group algebra associated with G. This so called regular representation is special since it contains all the irreducible representations of G. Irreducible representations are in bijective correspondence with the conjugacy classes of G.

Let us now focus on the case $G = S_n$. Each conjugacy class of S_n is indexed by a unique partition λ of n, where the components of λ determine the cycle structure of some member of the conjugacy class (choice of member does not affect answer).

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n, with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$. Then $|\lambda| := n$ denotes the size of λ, $\ell(\lambda) := k$ denotes its length and $\langle\lambda\rangle := \ell(\lambda) + \sum \log_2(\lambda_i)$ the space required to represent the partition. The Ferrers' diagram of λ i.e. F_λ is the set of left-justified boxes with λ_i squares in the i-th row (1st row = top row). By abuse of notation F_λ is also denoted λ.

Corresponding to each partition λ, we define a_λ to be the sum of those elements of S_n which leave all the rows of F_λ invariant (number the boxes in F_λ in the canonical order). Similarly b_λ is the signed sum of those elements of S_n which leave the columns of F_λ invariant. Finally, let $c_\lambda = a_\lambda b_\lambda$ be the Young's symmetrizer, corresponding to λ. Then the representation S_λ is the image of the endomorphism on $\mathbb{C}[S_n]$ which takes $x \mapsto x c_\lambda$.

Now before we come to the characters of these irreducible representation, some definitions are in order.

Definition 6.1 *Put $x = (x_1, \ldots, x_n), 0 \leq r \leq n, \lambda \vdash n, \mu \vdash n$.*

- *(r'th power symmetric function)* $P_r(x) = \sum_i x_i^r$,
- *(Power symmetric functions)* $P_\lambda(x) = \prod_j P_{\lambda_j}(x)$,
- *(Anti-symmetric polynomials)* $A_\lambda(X) = \det ||x_i^{\lambda_j + n - j}||$,
- *(Discriminant)* $D(x) = A_{(0,\ldots,0)}(x) = \prod_{i<j}(x_i - x_j)$,
- *(Schur polynomials)* $s_\lambda(x) = A_\lambda(x)/D(x)$,
- *(r'th Homogenous symmetric functions)* $h_r(x)$ = *sum of all distinct monomials of degree r, and*

- *(Homogenous symmetric functions)* $h_\lambda(x) = \prod_j h_{\lambda_j}(x)$

Note that since A_λ is anti-symmetric, s_λ is a polynomial with integer coefficients. Both s_λ as well as P_μ form a basis for the symmetric homogenous functions of degree n, as λ and μ range over partitions of n.

Let χ be class function on S_n (i.e. constant on conjugacy classes). Define the Frobenious map,

$$\mathcal{F}(\chi) = \frac{1}{n!}\sum_{\mu \vdash n} \chi(\mu) P_\mu(x) \tag{6.1}$$

where $\chi(\mu)$, represents the value of the character at the conjugacy class corresponding to μ. Every homogenous symmetric polynomial of degree n is of the form $\mathcal{F}(\chi)$ for a suitable character χ. This is easily seen as follows: Start with a symmetric homogenous polynomial f of degree n. Since the $\{P_\lambda\}$'s form a basis of this space, $f = c_\lambda P_\lambda$ for suitable scalars λ. Now define χ so that $\chi(\sigma) = n!/N_\lambda c_\lambda$, where λ corresponds to the conjugacy class of σ and N_λ is the number of elements in that conjugacy class. Since $n!/N_\lambda$ is always an integer, $\chi(\sigma)$ is always an integral multiple of c_λ.

The remarkable connection between the irreducible characters and the Schur polynomials is given by

$$s_\lambda(x) = \mathcal{F}(\chi^\lambda),$$

where χ^λ is the character corresponding to the irreducible representation S_λ of S_n. Hence the value of the irreducible characters of S_n are the coefficients which occur when the Schur polynomials are expressed in terms of the Power symmetric functions. This relation can be inverted, and similar coefficients appear when the Power symmetric functions are expressed in terms of the Schur polynomials.

Given two homogenous symmetric polynomials of degree n, f_1 and f_2, define their Kronecker product $f_1 \otimes f_2$ as $\mathcal{F}(\chi_1\chi_2)$, where $\chi_i = \mathcal{F}^{-1}(f_i)$, for $i = 1, 2$. This definition is motivated by the following observation: With this definition, we have that $s_\lambda \otimes s_\mu$ corresponds to the character $\lambda\mu$ of the representation $S_\lambda \otimes S_\mu$. Moreover if $s_\lambda \otimes s_\mu = g_{\lambda\mu\nu} s_\nu$, then $g_{\lambda\mu\nu}$ gives the multiplicity of the representation S_ν in $S_\lambda \otimes S_\mu$. The reasoning is as follows:

If $S_\lambda \otimes S_\mu = \sum g_{\lambda\mu\nu} S_\nu$, then we also have $\chi^\lambda \chi^\mu = \sum g_{\lambda\mu\nu}\chi^\nu$. Then by linearity of $\mathcal{F}$, we have $\mathcal{F}(\chi^\lambda\chi^\mu) = \sum g_{\lambda\mu\nu}\mathcal{F}(\chi^\nu)$, i.e. $s_\lambda \otimes s_\mu = \sum g_{\lambda\mu\nu} s_\nu$ as promised. Moreover, if $(\ ,\ )$ denotes the Hall inner product on symmetric functions, then we also have that $g_{\lambda\mu\nu} = (s_\lambda \otimes s_\mu, s_\nu)$.

6.2 Basic Definitions

In the rest of this exposition, we explore the values of $g_{\lambda\mu\nu}$ when λ and μ are restricted to a small class of shapes.

Definition 6.2 *Let $0 \le t \le n$. Then the partition $(1^t, n-t)$ is said to be a **hook shape** and the partition $(t, n-t)$ is said to be a **two-row shape**.*

It was known before that for partitions $\lambda \vdash n, \mu \vdash n$ and $\nu \vdash n$, such that λ and μ are hook shapes, then $\forall \nu\, g_{\lambda\mu\nu} \le 2$. Similarly, if λ is a hook shape and μ is a two-row shape, then $\forall \nu\, g_{\lambda\mu\nu} \le 3$. Both results hold for an arbitrary n. This is the first result, where $g_{\lambda\mu\nu}$ can be unbounded.

If λ and μ are partitions of n then we write $\lambda \leq \mu$ if $\ell(\lambda) \leq \ell(\mu)$ and $\lambda_{\ell(\lambda)-p} \leq \mu_{\ell(\mu)-p}$ for $0 \leq p \leq \ell(\lambda)$. In other words F_λ is completely inside F_μ, when they are superposed with the bottom left corner box aligned. If $\lambda \leq \mu$, denote by $F_{\mu/\lambda}$ the diagram of the skew shape μ/λ obtained by removing the boxes corresponding to λ from F_μ.

Let $\lambda \vdash n$ and $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a sequence of integers such that $\sum \alpha_i = n$. A **decomposition** of λ of type α denoted by $D_1 + \cdots + D_k = \lambda$, is a sequence of shapes

$$\lambda^1 \subset \lambda^2 \subset \cdots \subset \lambda^k = \lambda$$

$D_i = \lambda^i/\lambda^{i-1}$ a skew shape and $|D_i| = \alpha_i$.

A **column strict tableau** T of shape μ/λ is a filling of $F_{\mu/\lambda}$ with positive integers such that in each row, the numbers weakly increase from left to right, and in each column strictly increase from bottom to top. T is said to be **standard** if the entries of T are precisely $1, \ldots, |T| = |\mu/\lambda|$. Let $CS(\mu/\lambda)$ denote the set of all column strict tableau's of shape μ/λ and $ST(\mu/\lambda)$ the set of all standard tableau's of the same shape. For $T \in CS(\mu/\lambda)$, denote by $w(T)$ the monomial obtained by replacing $i \in T$ with x_i and taking the product over all boxes, i.e. $w(T) = \prod_{i=1}^{|T|} x_i^{N_i}$, where $N(i)$ is the number of occurrences of i in T.

Now, the skew Schur function $s_{\mu/\lambda}$ is defined by

$$s_{\mu/\lambda}(x_1, x_2, \ldots) = \sum_{T \in CS(\mu/\lambda)} w(T) \tag{6.2}$$

Setting $\lambda = \emptyset$ we get a combinatorial definition of the usual Schur functions. When $\lambda = \emptyset$, we call the partition μ/λ a straight shape, to distinguish it from the more general skew shape. The n'th homogenous symmetric function h_n is thus defined by $h_n = s_{(n)}$.

A skew shape μ/λ is said to be a **horizontal r-strip**, if $|\mu/\lambda| = r$ and no two boxes of μ/λ are in the same column.

For two shapes λ and μ, denote by $\lambda \star \mu$ the skew diagram obtained by joining at the corners the rightmost lowest box of F_λ to the leftmost highest box of F_μ.

Hence we have that $CS(\lambda \star \mu) = CS(\lambda) \times CS(\mu)$, and this in turn implies that $s_\lambda s_\mu = s_{\lambda\star\mu}$. So, we can write an arbitrary product of Schur functions (of straight shapes) as the Schur function of a single skew shape.

6.3 Basic Formulae

We start with some basic properties of the Kronecker product.

$$\begin{array}{rcl} h_n \otimes s_\lambda & = & s_\lambda, \text{ i.e. } \mathcal{F}(\text{trivial char}) = h_n \\ s_{(1^n)} \otimes s_\lambda & = & s_{\lambda'}, \text{ where } \lambda' \text{ is the conjugate partition of } \lambda \\ s_\lambda \otimes s_\mu & = & s_\mu \otimes s_\lambda = s_{\lambda'} \otimes s_{\mu'} = s_{\mu'} \otimes s_{\lambda'} \\ (P+Q) \otimes R & = & P \otimes R + Q \otimes R \\ g_{\lambda_1\lambda_2\lambda_3} & = & g_{\lambda_{\pi(1)}\lambda_{\pi(2)}\lambda_{\pi(3)}}, \text{ for any permutation } \pi \in S_3. \end{array}$$

Fact 6.3 *Some results about Schur functions*

- ***(Pieri's Rule)*** $h_r \cdot s_\lambda = \sum_\mu s_\mu$, *where the sum is over all μ such that μ/λ is a horizontal r-strip.*

- ***(Jacobi-Trudi identity)*** $s_\lambda = \det ||h_{\lambda_j - i + j}||_{1 \le i,j \le \ell(\lambda)}$, *where* $h_0 = 1$ *and* $h_r = 0$ *for* $r < 0$.
- ***(Littlewood)*** $s_\alpha s_\beta \otimes s_\lambda = \sum_{\gamma \vdash |\alpha|} \sum_{\delta \vdash |\beta|} c_{\gamma\delta\lambda}(s_\alpha \otimes s_\gamma)(s_\beta \otimes s_\delta)$, *where the* $c_{\gamma\delta\lambda}$ *are the Littlewood-Richardson coefficients, i.e.* $c_{\gamma\delta\lambda} = (s_\gamma s_\delta, s_\lambda)$.

Some of the above proofs can be found in [FH, Appendix A]. Pieri's rule gives us a way to multiply Schur polynomials with a homogenous symmetric function, and the Jacobi-Trudi identity allows us to express Schur polynomials in terms of homogenous symmetric functions. One simple consequence of the Jacobi-Trudi identity is that $s_{(k)} = h_k$, i.e. if the partition has only one part, the Schur polynomial is the same as the corresponding homogenous symmetric function. Littlewood's result was used by Garsia and Remmel to show that

$$(s_H \cdot s_K) \otimes s_D = \sum_{\substack{D_1 + D_2 = D \\ |D_1| = |H| \\ |D_2| = |K|}} (s_H \otimes s_{D_1}) \cdot (s_K \otimes s_{D_2})$$

which by induction can be used to show that

$$(h_{a_1} \dots h_{a_k}) \otimes s_D = \sum_{\substack{D_1 + \dots + D_k = D \\ |D_i| = a_i}} s_{D_1} \dots s_{D_k} \tag{6.3}$$

Note that the right hand side of the above expression does not contain any $\otimes$ and the result from which it was proved contains $\otimes$. This is possible because in our case, the Schur functions involved are just those with exactly one part. Hence they are just h_k for suitable k, and we know that $h_n \otimes s_\lambda = s_\lambda$.

A naïve upper bound for the number of terms which occur in the above expansion is the number of ways in which $|D|$ can be partitioned into k sets with the i'th set having a_i elements, which is the multi-nomial coefficient $\binom{|D|}{a_1 \dots a_k}$. Note that the right hand size vanishes if $|D| \neq \sum a_i$.

6.4 A naïve algorithm

At this point we know enough to give an algorithm to calculate the tensor product of two Schur functions, though it is horribly inefficient.

Input: Partitions λ, μ of n.
Output: $\{g_{\lambda\mu\nu}\}$, i.e. number of times each representation S_ν occurs in $S_\lambda \otimes S_\mu$.
Notation: Let $l := \ell(\lambda), m := \ell(\mu)$.

1. Using the Jacobi-Trudi identity, write s_λ as a polynomial in the basic homogenous symmetric functions. The number of terms in the expansion is $l!$.
2. Similarly write s_μ as a polynomial in the basic homogenous symmetric functions. The number of terms in the expansion is $m!$.
3. Using $(P + Q) \otimes R = (P \otimes R) + (Q \otimes R)$, we can write $s_\lambda \otimes s_\mu$ as the sum of $l!m!$ terms, each of which is of the form $s_{a_1} \dots s_{a_l} \otimes s_{b_1} \dots s_{b_m}$.

4. To compute such a term, use equation (6.3) and reduce it to computing $s_{D_1} \dots s_{D_l}$ over decompositions $D_1 + \cdots + D_l = D := (b_1) \star \cdots \star (b_m)$, with $|D_i| = a_i$. Number of such terms is bounded by $\binom{\sum b_j}{a_1 \dots a_l} = \binom{n}{a_1 \dots a_l}$, since $\sum b_j = |\mu| = n$.

5. Consider a typical term $s_{D_1} \dots s_{D_l}$. Each skew shape D_i is a "subshape" of the skew shape D. Hence each of the skew Schur function's s_{D_i} is a simple product of Schur functions with one part, i.e. homogenous symmetric functions. Each typical term becomes a product of $O(n)$ homogenous symmetric functions.

6. **At this point we have eliminated the $\otimes$ in favor of multiplication**

7. Now we can use the Pieri rule to write $h_a \cdot h_b$ as a sum of Schur functions. Note that if $h_a \cdot h_b = \sum_\mu s_\mu$, then μ can have at most two parts. If a and b are both less than n, then the number of different μ's which can contribute is $O(n)$.

8. Repeating the above step $O(n)$ times we can calculate each term $s_{D_1} \dots s_{D_l}$.

9. Now collecting all the terms, we get the complete decomposition of $s_\alpha \otimes s_\beta$.

The step of the algorithm where we use equation (6.3) contributes $\binom{n}{a_1 \dots a_l}$, which can be as bad as $l^n / \binom{n+1}{l-1}$ (by Pigeon Hole Principle). This step alone makes this algorithm worse than the one we get from character theory for large l, unless we can reduce the number of terms we need to deal with by a lot.

The total running time for this algorithm is $O(l! \cdot m! \cdot l^n \cdot ((l-1)/n)^{(l-1)} \cdot n^2)$, assuming all polynomial operations can be done in unit time (which is not true but its cost is only $2^O(n)$ which is dominated by the l^n any way). If one is only interested in the coefficient of one particular s_ν, one may be able to improve on the running time.

Conjecture: The decision problem, which given λ, μ, ν tells if $g_{\lambda\mu\nu} > 0$ is polynomial computable in the input length, i.e. in $\langle\lambda\rangle + \langle\mu\rangle + \langle\nu\rangle$.

6.5 Outline of the proof

Now we are in a position to give an outline of the proof of the main result.

$s_{(h,k)} \otimes s_{(l,m)} = \sum_\nu g_{(h,k)(l,m)\nu} s_\nu$, and $g_{(h,k)(l,m)\nu} = 0$ if ν has more than 4 parts. Otherwise let $\nu = (a, b, c, d)$. Then $g_{(h,k)(l,m)(a,b,c,d)}$ is given by a sum of 8 terms each of the form $\sum_{r=L}^{U} 1 + \min(expr_1, expr_2)$, where each L is in turn given by a max of 2 to 5 terms and each U is given by a min of 2 to 5 terms.

Even though this formula is messy, it is the first formula for $g_{\lambda\mu\nu}$ at least in special cases. We know proceed with the outline of the proof. The proof basically, follows the naïve algorithm of the previous section with a few tricks thrown in.

From the Jacobi-Trudi identity, we have

$$s_{(h,k)} = \det \begin{pmatrix} s_h & s_{k+1} \\ s_{h-1} & s_k \end{pmatrix} = s_h s_k - s_{h-1} s_{k+1}$$

and similarly for $s_{(l,m)}$. Hence we have,

$$\begin{aligned} s_{(h,k)} \otimes s_{(l,m)} &= (s_h s_k - s_{h-1}s_{k+1}) \otimes (s_l s_m - s_{l-1}s_{m+1}) \\ &= s_h s_k \otimes s_l s_m - s_h s_k \otimes s_{l-1}s_{m+1} \\ &\quad - s_{h-1}s_{k+1} \otimes s_l s_m + s_{h-1}s_{k+1} \otimes s_{l-1}s_{m+1} \end{aligned}$$

Let $A = s_h s_k \otimes s_l s_m$, $B = s_h s_k \otimes s_{l-1}s_{m+1}$, $C = s_{h-1}s_{k+1} \otimes s_l s_m$ and $D = s_{h-1}s_{k+1} \otimes s_{l-1}s_{m+1}$, so that $s_{(h,k)} \otimes s_{(l,m)} = A - B - C + D$.

Now the expansion of A, can be obtained from equation (6.3) by forming all decompositions $D_1 + D_2 = (l) \star (m)$ with $|D_1| = h$ and $|D_2| = k$, each term of which can then be evaluated by repeated applications of the Perri rule.

Similar treatment can be given to the other terms as well. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ represent the set of configurations which occur in the expansion of the corresponding term. The main observation, is that one can define a map I on $\mathbf{A} \cup \mathbf{B} \cup \mathbf{C} \cup \mathbf{D}$ to itself such that $I^2 = Id$. Moreover the map is such that for any configuration S for which $I(S) \neq S$, the terms in the expansion of $s_{(h,k)} \otimes s_{(l,m)}$ for S and $I(S)$ cancel each other out. Hence it will be enough to look at those configurations for which $I(S) = S$.

References

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